Biyani's Think Tank

Concept based notes

Differential Equation

B.Sc. Part-II Year

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this endeavour. They played an active role in coordinating the various stages of this endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Syllabus

Unit 1: Degree and order of a differential equation. Equations of first order and first degree. Equations in which the variables are separable. Homogeneous equations and equations reducible to homogeneous form. Linear equations and equations reducible to linear form. Exact differential equations and equations which can be made exact.

Unit 2: First order but higher degree differential equations solvable for x,y and p. Clairaut's form and singular solutions with Extraneous Loci. Linear differential equations with constant coefficients, Complimentary function and Particular integral.

Unit 3: Homogeneous linear differential equations, Simultaneous differential equations. Exact linear differential equations of nth order. Existence and uniqueness theorem.

Unit 4: Linear differential equations of second order. Linear independence of solutions. Solution by transformation of the equation by changing the dependent variable/independent variable, Factorization of operators, Method of variation of parameters, Method of undetermined coefficients.

Unit 5: Partial differential equations of the first order. Lagrange's linear equation. Charpit's general method of solution. Homogeneous and non-homogeneous linear partial differential equations with constant coefficients. Equations reducible to equations with constant coefficients.

(UNIT-I)

Differential Equation of first order and first degree : Homogeneous equations, Linear Equations and Exact differential Equation

Differential Equation : An equation involving a function and its derivatives is called a differential equation for eg.

1. $\frac{dy}{dx}+\frac{3}{x}=x^2$
2. $(x^2+y^2)dx-2xydy=0$

Order and degree of differential equation:

- **ORDER:-** The order of a differential equation is the order of highest order derivative appearing in the equation.

  For eg: $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^x$

  Is a differential equation of second order.
- **DEGREE:-** The degree of a differential equation is the degree of the highest order derivative, when the differential coefficients are free from radicals and fractions.

  For eg :

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$\Rightarrow \rho^2\left(\frac{d^2y}{dx^2}\right)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

  This is a differential equation of the second order and second degree.

- **Equations of first order and first degree :-**

  Standard from :- Standard form of the equation of the first order and first degree can be written as –

$$f_1(x,y)dx+f_2(x,y)dy=0$$

  Or Mdx+Ndy=0

  Where $f_1(x,y)$ and $f_2\ (x,y)$ or M and N are functions of x and y.
- **CASE-I :- Equation in which the variable are separable** :- Differential equation of following forms may be considered under this category

$f_1(x)$dx+ f_2 (y)dy= 0 ....(1)

Equation 1- can also be written as $f_1(x) + f_2(y)\frac{dy}{dx} = 0$

Which on integration with respect to x give:-

$\int f_1(x)dx + \int f_2(y)\frac{dy}{dx}dx = C$

Or $\int f_1(x)dx + \int f_2(y)dy = C$ ....(2)

Where C is any arbitrary constant of integration.
Of integration
Eq(2) is general solution of Eq(1)

- **CASE-II :- When equation are Homogenous :**

A differential equation of the form

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$$

Where $f_1(x,y)$ and $f_2(x,y)$ are Homogeneous function of same degree, is known as a Homogenous differential equation. Such an equation can be transformed into an equation in which the variables are separated by the substitution $y = vx$ (or $X = vy$, where v is new variable

- **Reason why the substitution y= vx transforms the equation into one in which the variables are separable :-**

The reason the substitution y = vx transform the equation in to one in which the variables are separable can be seen when the given equation is written in the form

$\frac{dy}{dx} = \frac{M(x_1y)}{N(x_1y)}$ ....(1)

If $M(x,y)$and $N(x,y)$ are homogenous function of the same degree and one substitutes vx for y one finds that X's all cancel out on the right side of eq(1) and the right side becomes a function in V alone i.e. the equation takes the form

$\frac{dy}{dx} = g(v)$ ....(2)

Substituting dy= v dx+xd v then gives

$vdx + xdv = g(v)dx$ ....(3)

Where the variables can be separated as

$$\frac{dv}{g(v)-v} = \frac{dx}{x}$$

- **CASE-III :- When equation are linear**:- An equation of the form

$\frac{dy}{dx} + PY = Q(x)$ ....(1)

Where P and Q are function of x(or constant) is called linear equation of first order.

→ a differential equation is linear in which the dependent variable and its derivative occur in first degree.

Linear equations are solved when they are

Multiplied by $e^{\int p dx}$ which is called integrating factor (I.F.)because by the multiplication of this factor the left side of eq(1) become perfect.

The general solution is given by

$$ye^{\int p dx} = \int e^{\int p dx} . Q dx + c$$

or Y.I.F. = $\int I.F. Q. dx + c$

- **CASE-IV - : Bernoulli's Differential equations:-**

The equation

$\frac{dy}{dx} + P(x)y = Q(x)y^n$ ....(1)

Is known as Bernoulli's equation. It can be transformed into a linear equation by the transformation $y^{-n+1} = v$ ....(2)

where v is a new variable.

Let us divide eq.(1) by y^n to obtain the equivalent equation.

$y^{-n} \frac{dy}{dx} + y^{-n+1} P(x) = Q(x)$ ....(3)

Now take the derivative of eq(2) with respect to x, we obtain

$y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dv}{dx}$ ....(4)

Substituting e q. (2) and (4) into eq(3), yield $\frac{1}{1-n} \frac{dv}{dx} + vp(x) = Q(x)$

Or

$\frac{dv}{dx} + v \ [(1-n) \ P(x)] = (1-n)Q(x)$

Which is a linear equation in the variable v. The general solution for Bernoulli's equation is

$$ye^{\int p dx} = \left[(1-n) \int Q(x) \quad e^{(1-n)p(x)dx} \ dx + c\right]^{\frac{-1}{n}}$$

- **CASE-V:- When equation are exact:-** A differential equation which has been formed from its primitive by differentiation and without any further operation of elimination or reduction is said to be exact.

  The necessary and sufficient condition for the equation Mdx+Ndy=0 to an exact differential equation is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

  If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is true, then we have to show that Mdx+Ndy=0 will be an exact differential equation. The solution of the exact differential equation

  $= Mdx + Ndy = 0 \text{ is given by}$

  $\int (Mdx + Ndy) = \int d[P + f(y)] = C$

  OR $P(x,y) + f(y) = C$

  OR $P(x,y) = \int Mdx[\text{integrate } M\ w_0.r.t.\ x \text{ regarding } y \text{ as constant}]$

  And f(y)= $\int \left(N - \frac{\partial P}{\partial Y} \right) dy$

- **CASE –VI:- Equation reducible to an exact differential equation .**

  A differential equation of the type Mdx+Ndy=0 which is not exact can sometimes become exact by multiplying the equation by some function of x and y, which is called the integrating factor.

Rule –I:- Integrating factor found by Inspection:-

Sno	Term	Integration factor	Exact differential
1	$xdy - ydx$	(i) $\frac{1}{x^2}$ (ii) $\frac{1}{y^2}$ (iii) $\frac{1}{xy}$ (iv) $\frac{1}{x^2+y^2}$ (v) $\frac{1}{x^2+y^2}$ (vi) $\frac{1}{x^2-y^2}$	$\frac{xdy-ydx}{x^2} = d\left(\frac{y}{x}\right)$ $\frac{-ydx-xdy}{y^2} = -d\left(\frac{y}{x}\right)$ $\frac{xdy-ydx}{xy} = d\log\left(\frac{y}{x}\right)$ $\frac{xdt-ydx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ $\frac{-ydx-xdy}{x^2+y^2} = -d\left(\tan^{-1}\frac{x}{y}\right)$ $\frac{xdy-ydx}{x^2-y^2} = d\left\{\frac{1}{2}\log\frac{x-y}{x-y}\right\}$
2	$Xdy + ydx$	(i) $\frac{1}{(xy)^n}$ (ii) 1 (iii) $\frac{1}{\sqrt{1-x^2y^2}}$	$\frac{xdy+ydx}{(xy)^n} = d\left\{\frac{-1}{(n-1)(xy)^{n-1}}\right\}$ $n\#1$ =dlog(x+y) n=1 $= xdy + ydx = d(xy)$ $\frac{xdy+ydx}{\sqrt{1-x^2y^2}} = d[(\sin^{-1}(xy)]$

3	$xdx + ydy$	$\frac{1}{(x^2+y^2)^n}$	$\frac{xdx+ydy}{(x^2-y^2)^n} = d\left\{\frac{-1}{2(n-1)(x^2-y^2)^{n-1}}\right\}$ $n \neq 1$ $\frac{xdx+ydy}{x^2-y^2} = d\left\{\frac{1}{2}\log(x^2+y^2)\right\}$ $when\ n = 1$
	$y^2dx + 2xydy$	$\frac{1}{x^2y^4}$	$\frac{y^2dx+2xydy}{x^2y^4} = d\left(\frac{-1}{xy^2}\right)$
	$2xy\,dy - y^2dx$	$\frac{1}{x^2}$	$\frac{2xydy - y^2dx}{x^2} = d\left(\frac{y^2}{x}\right)$
	$2xy\,dx - x^2dy$	$\frac{1}{y^2}$	$\frac{2xydx - x^2dy}{y^2} = d\left(\frac{x^2}{y}\right)$
	$2xy^2dx - 2x^2ydy$	$\frac{1}{y^4}$	$\frac{2xy^2dx - 2x^2ydy}{y^4} = d\left(\frac{x^2}{y^2}\right)$
	$xe^ydy - e^ydx$	$\frac{1}{x^2}$	$\frac{xe^ydy - e^ydx}{x^2} = d\left(\frac{e^y}{x}\right)$
	$ye^xdx - e^xdy$	$\frac{1}{y^2}$	$\frac{ye^xdx - e^xdy}{y^2} = d\left(\frac{e^x}{y}\right)$

- **Rule:-2** if the equation Mdx+Ndy=0 is homogeneous and Mx+Ny$\neq 0$ then the integration factor may be $\frac{1}{Mx+Ny}$
- **Rule:3-** if the equation Mdx+Ndy=0 has the form $f_1(xy)ydx + f_2(xy)x\,dy = 0,$ *then* its one I.F. will be $\frac{1}{Mx-Ny}$ provided the denominator is not zero.
- **Rule:4 –** if in the equation Mdx+Ndy=0, the value of $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone $\{say\ f(x)\}$ then $\int^{f(x)dx}$ *will be the integrating factor of the equation*
- **Rule:5-** if in the equation Mdx+ Ndy=0, the value of $\frac{1}{M}\left(\frac{\partial N}{\partial X} - \frac{\partial M}{\partial Y}\right)$ is a function of y alone $\{say\ g(g)\}$ *then* $e^{\int g(y)dy}$ *will be the I.F. of the equation.*
- **Rule:-6 –** if the equation Mdx +Ndy=0 is of the form $x^p y^p\,(aydx + bxdy) + x^r y^s(\,lydx +$ *mxdy=0 where P,q,a,b,r,s,l,m are constant then xhyk will be determined from the* Condition that after their multiplication the equation will be exact.

Q-1 **(e^y+1) cosx dx + e^y sin x dy=0**

Sol: The given equation can be written as

Cot x dx $+\frac{e^y}{e^y+1}\ dy = 0$

On integrating we get

$$\int \cot x\ dx + \int \frac{e^y}{e^y+1}\ dy = c$$

Or log sin x + log $(e^y + 1)$= log c_1

Where C is an arbitrary constant of integration

$(e^y + 1)$sinx =C_1 is the required solution.

Q-2 $(x - y^2)dx + 2xy\ dy = 0$

Sol. Substituting $y^2 = z\ So\ that\ 2y\ dy = dz$

The equation can be written as

$$(x - z\ )dx + xdz = 0$$

Or $xdx - zdx + xdz = 0$

Or $\frac{dx}{x} + \left(\frac{xdz - zdx}{x^2}\right) = 0$

Integrating $\log x + \frac{z}{x} = constant$

Or $xe^{\frac{y^2}{x}} = A\ is\ the\ requried\ solution.$

Q-3 $\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}y - \left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\} x\frac{dy}{dx} = 0$

Sol The equation is homogenous of degree two and can be solved by the substitution $y = \quad vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ hence

$$\{x\cos v + vx\sin\ v\}vx - \{\ vx\sin v - x\cos v\}\ x\left(v + \frac{xdv}{dx}\right) = 0$$

Or $\{\cos v + v\sin\ v\}v - \{\ v\sin v - \cos v\}\ \left(v + \frac{xdv}{dx}\right) = 0$

Or $\frac{(\cos v + v\sin v)v}{v\sin v - \cos v} - \ v = \frac{xdv}{dx}$

Or $\frac{2v\cos v}{v\sin v - \cos v} \ = \frac{xdv}{dx}$

Or Separation of variable, we have

$$2\frac{dx}{x} = -\left(\frac{-v\sin v + \cos v}{v\cos v}\right)dv$$

Integrating

$$2\log x = -\log((v\cos v) + constant$$

Or $x^2 v\cos v = C$ (arbitrary constant of integration)

Or $xy\cos\frac{y}{x} = C$ is the required solution

Q-4 $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

Solu Substituting $x = vy$ so that $dx = vdy + ydv$ then

$(1+e^{v})(vdy + ydv) + e^{v}(1-v)dy = 0$

Or $\frac{1}{y}dy + \frac{1+e^{v}}{v+e^{v}}dv = 0$

Integrating $\log y + \log(v + e^{v}) = constant$

Or $y\,(v + e^{v}) = c$

Or $y\,\left(\frac{x}{y} + e^{x/y}\right) = c$

Hence the required solution is $x + ye^{x/y} = C$

Q-5 $\frac{dy}{dx} = \frac{(x+y-1)^2}{4(x-2)^2}$

Soul Putting $x = X + h$ and $y = Y + K$ in the given equation we have

$\frac{dy}{dx} = \frac{\{X+Y+(h+K-1)\}^2}{4\{X+(h-2)\}^2}$ ....(1)

choosing h and k such that $h + k-1=0$ and h-2=0 which gives h=2 and K=-1

hence from (i), we will have

$\frac{dy}{dx} = \frac{(X+Y)^2}{4x^2}$

Putting Y= vX so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$ we get

X $\frac{dv}{dx} = \frac{(1+v)^2}{4} - v = \frac{(v-1)^2}{4}$

Separating the variables

$$\frac{4dv}{(v-1)^2} = \frac{dX}{X}$$

Integrating $\frac{4}{1-v} = logX + \log C$

Or $\frac{4}{1-v} = \log\ CX\ or\ CX = e^{4/1-v}$

CX = exp $\frac{4x}{X-Y}$

Replacing X by (x -2), Y by (y +1), we get the required solution as

C(x -2) = $e^{4(x-2)/(x-y-3)}$

Q-6 $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Sol.: $\frac{dy}{dx} = e^{-y}(e^x + x^2)$

By separation of variable we have

$$e^y dy = (e^x + x^2)dx$$

Integrating $\int e^y dy = \int (e^x + x^2)dx$

or $e^y = e^x + \frac{x^3}{3} + c$ is the required solution

Q.7 $\frac{dy}{dx} = \frac{x+y+1}{x-y}$

Ans $\frac{dy}{dx} = \frac{x+y+1}{x-y}$ -------------(1)

Here $\frac{a}{A} \neq \frac{b}{B}$ $(\therefore\ 1 \neq -1)$

Therefore we suppose $x = X + h, y = Y + K$ -------------(2)

Then new form will be the given eg'^n (1)

$$\frac{dy}{dx} = \frac{X+Y+(h+k+1)}{x-y+(h-k)}$$ ------------- (3)

$h + k + 1 = 0$ and $h - k = 0$

$\therefore h = k = -1/2$ ------------(4)

Therefore from eq (3) $\frac{dy}{dx} = \frac{X+Y}{X-Y}$ which is Homogeneous equation substitute $y = vx$

$$\frac{dy}{dx} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = \frac{1+v}{1-v}$$

Or $X\frac{dv}{dX} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}$

Separating the variables

$$\frac{1-v}{1+v^2}\,dv = \frac{dX}{X}$$

Or $\left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \frac{dX}{X}$

After integrate we can write

$tan^{-1}v - \frac{1}{2}log\,(1+v^2) = log\,X + c$ -----------(5)

$\therefore\; v = \frac{y}{X} = \frac{y-\frac{1}{2}}{x-\frac{1}{2}} = \frac{2y+1}{2x+1}$ -------------(6)

And $X = x + \frac{1}{2} = \frac{2x+1}{2}$ -------------(7)

From Equation (5), (6), (7) we can write

$$tan^{-1}\left(\frac{2y+1}{2x+1}\right) - \frac{1}{2}log\left[1 + \left(\frac{2y+1}{2x+1}\right)^2\right]$$

$$= C + \log\left(\frac{2x+1}{2}\right)$$

Or $tan^{-1}\left(\frac{2y+1}{2x+1}\right) - \frac{1}{2}log\left(x^2 + y^2 + x + y + \frac{1}{2}\right) = C$

Q.8 **Solve $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$**

Sol. The given equation can be written in this way

$\frac{dy}{dx} = \frac{2x+3y+4}{2(2x+3y)+5}$ ------------(1)

Here $\frac{a}{A} = \frac{b}{B} = \frac{1}{2}$

Therefore suppose $W = 2x + 3y \Rightarrow \frac{dw}{dx} = 2 + 3\frac{dy}{dx}$

Then new form of the given equation will be

$\frac{1}{3}\left(\frac{dw}{dx} - 2\right) = \frac{W+4}{2W+5}$

Or $\frac{dw}{dx} = \frac{3w+12}{2w+5} + 2 = \frac{7w+22}{2w+5}$

By separation of variable

$$\left(\frac{2w+5}{7w+22}\right) dw = dx$$

Or $\frac{2}{7}\left[1 - \frac{9}{2}\left(\frac{1}{7w+22}\right)\right] dw = dx$

$$\therefore \int \left[1 - \frac{9}{2}\left(\frac{1}{7w+22}\right)\right] dw = \int \frac{7}{2}\, dx$$

$$or \quad w - \frac{9}{14}\log(7w+22) = \frac{7x}{2} + c$$

$$or \quad (2x+3y) - \frac{9}{14}\log(14x + 21y + 22) = \frac{7x}{2} + c$$

Which is the required solution of given equation.

Q-9 **Solve (x+2y³)** $\frac{dy}{dx} = y$

Sol. $\frac{dy}{dx} = \frac{y}{x-2y^3}$

Or $\frac{dx}{dy} = \frac{x+2y^3}{y}$ or $\frac{dx}{dy} - \frac{x}{y} = 2y^2$

Which is in the form of $\frac{dx}{dy} + Px = Q$

I.F.= $e^{\int -\frac{1}{y}dy} = e^{-\log y} = 1/y$

Therefore the required solution will be

$x.\frac{1}{y} = \int \frac{1}{y} . 2y^2 dy + C = y^2 + C$

$x = y(y^2 + c)$ *where c is any arbitary constant.*

Q-10 **Solve** $(xy^2 + x)dx + (yx^2 + y)dy = 0$

Sol $(xy^2 + x)dx + (yx^2 + y)dy = 0$

$x(y^2 + 1)dx + y(x^2 + 1)dy = 0$

By separation of variable , we obtain

$\frac{x}{x^2+1}\,dx + \frac{y}{y^2+1}\,dy = 0$

Integrating $\frac{1}{2}\log(x^2 + 1) + \frac{1}{2}\log(y^2 + 1) = \frac{1}{2}\log c$

Or $(x^2 + 1)(y^2 + 1) = C$ *is the required solution of given equation*

Q-11 **Solve** $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Sol. The given equation can be written

$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{2x \sin y \cos y}{\cos^2 y} = x^3$

$$or\ sec^2 y \frac{dy}{dx} + 2x\ tany = x^3 \qquad ------1$$

Or eq (1) is Bernoulli's equation from therefore we suppose

$$\tan y = v \Rightarrow sec^2\ y\frac{dy}{dx} = \frac{dv}{dx} \qquad ------2$$

From equation (1) and (2)

$$\frac{dv}{dx} + 2xv = x^3$$

It's a linear equation in v

I.F. $= e^{\int 2x\,dx} = e^{x^2}$

Therefore the required solution will be

$$v.e^{x^2} = \int x^3.e^{x^2}dx + C$$

$$= \int \frac{1}{2} te^t dt + C \qquad [t = x^2]$$

$$= \frac{1}{2}(t-1)e^t + c$$

$$= \frac{1}{2}(x^2-1)e^{x^2} + c$$

$$\therefore \tan y.e^{x^2} = \frac{1}{2}(x^2-1)e^{x^2} + c \qquad \text{[putting the value of } v\text{]}$$

$$tany = \frac{1}{2}(x^2-1) + ce^{-x^2}$$

Q.12 Find the differential equation whose general solution is $y = ax^2 + bx$

Sol. Given equation $y = ax^2 + bx$ --------(1)

Differentiating (1) with respect to x

$$\frac{dy}{dx} = 2ax + b \qquad ----(2)$$

Again differentiating with respect to x

$$\frac{d^2y}{dx^2} = 2a \qquad ----(3)$$

Eliminate a, b from the equation (1), (2), (3) the required solution will be

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Q.13 Find the order and degree of the following differential Equation.

$$\frac{d^1y}{dx^1} + 5\frac{dy}{dx} + \int ydx = x^3$$

Sol. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+\int y dx = x^3$

Or $\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}+y=3x^2$

Order$\rightarrow 3$ Degree $\rightarrow 1$

Q.14 **Solve $xdx+ydy=a^2\left(\frac{xdy-ydx}{x^2+y^2}\right)$**

Sol. The given can be written as

$\left\{x+\frac{a^2y}{x^2+y^2}\right\}dx+\left\{y-\frac{a^2x}{x^2+y^2}\right\}dy=0$ ------(1)

Here $M=x+\frac{a^2y}{x^2+y^2}$ $N=y-\frac{a^2x}{x^2+y^2}$

$\therefore \frac{\partial M}{\partial y}=\frac{a^2(x^2+y^2).1-y.2y}{(x^2+y^2)^2}=\frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$

And $\frac{\partial N}{\partial x}=\frac{-a^2(x^2-y^2).1-x.2x}{(x^2-y^2)^2}=a^2\frac{(x^2-y^2)}{(x^2-y^2)^2}$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Therefore the given differential equation is exact for get the solution

(i) $u(x,y)=\int Mdx=\int\left\{x+\frac{a^2y}{x^2+y^2}\right\}dx$ [taking y as constant]

Or $=\frac{(x^2)}{2}+a^2y.\frac{1}{y}\ tan^{-1}\frac{x}{y}$

$u(x,y)=\frac{x^2}{2}+a^2tan^{-1}\frac{x}{y}$

(ii) $\frac{\partial u}{\partial y}=a^2\frac{1}{1-(x/y)^2}\left(\frac{-x}{y^2}\right)=-a^2x/x^2+y^2$

(iii) $N-\frac{\partial u}{\partial y}=\left\{y-\frac{a^2x}{x^2-y^2}\right\}+\left\{\frac{a^2x}{x^2-y^2}\right\}=y$

$\therefore v(y)=\int\left\{N-\frac{\partial u}{\partial y}\right\}dy=\int ydy=\frac{y^2}{2}$

Therefore the general solution of the given equations will be $u(x,y)+v(y)=c$

Or $\frac{1}{2}x^2+a^2\ tan^{-1}\left(\frac{x}{y}\right)+\frac{1}{2}y^2=C$

Or $x^2+2a^2tan^{-1}\left(\frac{x}{y}\right)+y^2=k$

Where K is any constant.

Q. 15 **Solve (1+yx) xdy + (1-yx)$ydx = 0$**

Sol. The given equation can be written as $(xdy + ydx) + xy(xdy - ydx)$=0

By inspection its I.F. will be $\frac{1}{x^2y^2}$

$\therefore \frac{xdy+ydx}{(xy)^2} + \frac{xdy-ydx}{xy} = 0$

On integrating we obtain

$\frac{-1}{xy}$+ log $\left(\frac{y}{x}\right) = C$ which is the

Required solution of given equation

Q. 16 **Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy$=**

Sol. The given equation is Homogeneous equation of third degree.

Here $Mx + Ny$= $(x^2y - 2xy^2)x - (x^3 - 3x^2y)y$

= x^2y^2 #0

Therefore I.F. will be $\frac{1}{x^2y^2}$

Multiply the given equation by I.F.

$\left(\frac{x^2y-2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3-3x^2y}{x^2y^2}\right)dy = 0$

Or $\left(\frac{1}{y} \frac{-2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$ ------ (1)

$M = \frac{1}{y} \quad \frac{-2}{x}$ N= $\frac{-x}{y^2} + \frac{3}{y}$

$\frac{\partial M}{\partial y} = \frac{-1}{y^2}, \frac{\partial N}{\partial x} = \frac{-1}{y^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ therefore Eq. (1) is exact equation

On Integrating we obtain

$\frac{x}{y} - 2logx + 3logy = C$

Which is the required solution of the given equation.

Q. 17 **Solve $(x^3 + xy^4)dx + 2y^3 dy = 0$**

Sol. The given equation is in the form of $Mdx + Ndy = 0$

Here M = $x^3 + xy^4$ N= $2y^3$

$\frac{\partial M}{\partial y} = 4xy^3$ $\frac{\partial N}{\partial x} = 0$

$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y^3}(4xy^3 - 0) = 2x$ (function of only x)

Therefore I.F. = $e^{\int f(x)dx} = e^{\int 2x\,dx} = e^{x^2}$

Multiply the given equation by I.F.

$e^{x^2}(x^3 + xy^4)dx + 2e^{x^2}y^3 dy = 0$ ----------(1)

Here $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ therefore Eq. (1) is exact equation.

On Integrating we obtain

$\frac{e^{x^2}}{2}(x^2 + y^4 - 1)$ =C which is the required solution of the given equation.

Q. 18 $(xy^2 - x^2)dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$

Sol. Here M $= xy^2 - x^2 \Rightarrow (\partial M/\partial y) = 2xy$

and N= $3x^2y^2 + x^2y - 2x^3 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$

$\therefore \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= \frac{1}{xy^2 - x^2}[6xy^2 + 2xy - 6x^2 - 2xy] = 6$

Therefore I.F. = $e^{\int 6dy} = e^{6y}$

Multiply the given equation by I.F we obtain

$e^{6y}(xy^2 - x^2)dx + e^{6y}(3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$

Which is exact differential equation. On Integrating we obtain

$$e^{6y}\left(\frac{x^2y^2}{2}-\frac{x^3}{3}\right)+\frac{y^2e^{6y}}{6}-\frac{ye^{6y}}{18}+\frac{e^{6y}}{108}=C$$

$$e^{6y}\left(\frac{x^2y^2}{2}-\frac{x^3}{3}+\frac{y^2}{6}-\frac{y}{18}+\frac{1}{108}\right.$$

Which is the required solution of the given equation

Q. 19 **Solve $(x^2y^2+xy+1)ydx+(x^2y^2-xy+1)xdy=0$**

Solution Here M $=(x^2y^2+xy+1)$y, N= $(x^2y^2-xy+1)x$

$\therefore \frac{\partial M}{\partial y}$ # $\frac{\partial N}{\partial x}$ hence the given equation is not an exact differential equation. But M and N has the forms $f_1(xy)y$ and $f_2(xy)x$ correspondingly, so the integrating factor will be

$$\frac{1}{Mx-Ny}=\frac{1}{x^3y^3+x^2y^2+xy-x^3y^3+x^2y^2-xy}=\frac{1}{2x^2y^2}$$

Multiplying the given differential equation by $\frac{1}{2x^2y^2}$ we obtain

$$\frac{1}{2}\left(1+\frac{1}{xy}+\frac{1}{x^2y^2}\right)ydx+\frac{1}{2}\left(1-\frac{1}{xy}+\frac{1}{x^2y^2}\right)xdy=0$$

Which is now an exact differential equation and can be written as-

$$(ydx+xdy)+\left(\frac{1}{x}dx-\frac{1}{y}dy\right)+\left(\frac{1}{x^2y}dx+\frac{1}{xy^2}dy\right)=0$$

Or $d(xy)+d\left(log\frac{x}{y}\right)+\frac{d(yx)}{x^2y^2}=0$

Which on integration gives

$$xy+log\frac{x}{y}-\frac{1}{xy}=C$$

Q.20 **Solve $cos^2x\frac{dy}{dx}+y=tanx$**

Sol. The given Equation can be written as

$$\frac{dy}{dx}+ysec^2x=tanx.sec^2x$$

I.F. $=e^{\int sec^2x\,dx}=e^{tanx}$

Multiplication of the I.F. yields

$e^{tanx}\left[\frac{dy}{dx}+ysec^2x\right]=e^{tanx}\ sec^2x\ tanx$

or $\frac{d}{dx}[ye^{tanx}]=tanx\ sec^2x\ e^{tanx}$ which on integration gives

$ye^{tanx}=\int tanx\ sec^2x\ e^{tanx}\,dx+c$

or ye^{tanx} =c+ $e^{tanx}(tanx-1)$

Q.21 $(3xy-2ay^2)dx+(x^2-2ayx)dy=0$

Sol. $M=3xy-2ay^2$ $\frac{\partial M}{\partial y}=3x-4ay$

N= x^2-2ayx $\frac{\partial N}{\partial x}=2x-2ay$

so $\frac{\partial M}{\partial Y}$ # $\frac{\partial N}{\partial x}$

but $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{x(x-2ay)}\times(x-2ay)$

= $\frac{1}{x}$ (which is a fn. of x alone)

Therefore the I.F. = $e^{\int\frac{1}{x}dx}$

= e^{logx} = x

Multiplying the $eq.\ by x$

$(3x^2y-2axy^2)dx+(x^3-2ax^2y)dy=0$

which on integration provides

$(3x^2y\,dx+x^3dy)-2(axy^2dx+ax^2ydy)$

= $d\,(x^3y)-d(ax^2y^2)=0$

by integration

$x^3y-\ ax^2y^2=c$

Q.22 $(1-x^2)\frac{dy}{dx}+xy=xy^2$

Sol. $\frac{dy}{dx}+\frac{xy}{(1-x^2)}=\frac{xy^2}{(1-x^2)}$

$\frac{1}{y^2}\frac{dy}{dx}+\frac{x}{y(1-x^2)}=\frac{x}{(1-x^2)}$ ------------1

Taking $\frac{1}{y}=v$

$-\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

so by eq. (1)

$-\frac{dv}{dx}+\frac{x}{(1-x^2)}=\frac{x}{(1-x^2)}$

$\frac{dv}{dx}-\frac{x\ v}{(1-x^2)}=\frac{-x}{(1-x^2)}$

integrating factor I-F = $e^{-\int\frac{x}{(1-x^2)}dx}$

= $e^{\frac{1}{2}\int\frac{1}{t}dt}$

= $e^{\frac{1}{2}Log\ t}$

= $e^{Log\ (1-x^2)^{1/2}}$

I.F = $(1-x^2)^{1/2}$ Therefore the required solution will be

$v.(1-x^2)^{1/2}$ = $-\int\frac{x}{(1-x^2)}\ (1-x^2)^{1/2}$. dx

$v.(1-x^2)^{1/2}$ = $\int x\ \left(\sqrt{(1-x^2)}\right)^{-1}$. dx

$\frac{1}{y}(1-x^2)^{1/2}$ = $\frac{1}{2}\int(\sqrt{t})^{-1}.\,dt$

$\frac{1}{y}(1-x^2)^{1/2}$ = $\frac{1}{2}\int\left(\frac{1}{\sqrt{t}}\right).\,dt$

$\frac{1}{y}(1-x^2)^{1/2}$ = $\frac{1}{2}\ \frac{(t)^{\frac{1}{2}}}{1/2}+c$

$\frac{1}{y}(1-x^2)^{1/2}$ = $(1-x^2)^{1/2}+C$

$(1-x^2)^{1/2}-(1-x^2)^{1/2}=cy$

$(1-x^2)^{1/2} \quad (1-y)$= Cy

Q. 23 $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

Sol. Taking $x + y = t$

$1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\because \frac{dy}{dx} = \frac{dt}{dx} - 1$

Or $\frac{dt}{dx} - 1 = \frac{t+1}{2t+3}$

$\frac{dt}{dx} = \frac{t+1}{2t+3} + 1$

$\frac{dt}{dx} = \frac{t+1+2t+3}{2t+3}$

$\frac{dt}{dx} = \frac{3t+4}{2t+3}$

Separation of variables

$dx = \frac{2t+3}{3t+4} dt$

$dx = \left(\frac{2}{3} + \frac{1}{3}\frac{1}{(3t+4)}\right) dt$

By integration

$\int dx = \int \frac{2}{3} + \frac{1}{3} \frac{1}{(3t+4)} dt$

$x = \int \frac{2}{3} dt + \frac{1}{3} \int \frac{1}{(3t+4)} dt$

$x = \frac{2}{3} t + \frac{1}{9} \log (3t + 4) + c$

$x = \frac{2}{3}(x + y) + \frac{1}{9}\log (3x + 3y + 4) + \log c$

$9x = 6(x + y) + \log(3x + 3y + 4) + logc$

$3x - 6y = \log c \ (3x + 3y + 4)$

$e^{(3x-6y)} = c\,(3x + 3y + 4)$

$c\,(3x + 3y + 4) = e^{3\,(x-2y)}$

Multipule Choice Questions

Q.1 ***The order of the following differential equation is*** $(y'')^3 + (y')^4 + y^3 = 5x$

(A) 1 *(B)* 2 *(C)* 3 *(D)* 4

Answer(B)

Q.2 ***Consider the following differential equations (a)*** $y' = (\sin x)y + x^2$

(b) $y' = x(\sin y) + e^x$ ***(c)*** $y' = y^2 + x$

Which of the following statement is correct

(A) All the equations are linear
(B) (a) and (b) are linear
(C) (b) and (c) are linear
(D) Only (a) is linear

Answer (D)

Q.3 ***The order of the differential Equations*** $5\left(\frac{d^3y}{dx^3}\right)^5 + 6\left(\frac{d^1y}{dx^1}\right)^6 + 7y = 8$ ***is***

(A) 2 *(B)* 3 *(C)* 5 *(D)* 6

Answer (B)

Q.4 ***Which of the following differential equations are homogeneous ?***

(A) $y' = \frac{y+x}{x}$ *(B)* $y' = \frac{y^2}{x}$ *(C)* $y' \frac{x^2+y^2}{x^8}$

(i) *All (A), (B), (C)* *(ii) only (A)* *(iii) only (B)* *(iv) only (c)*

Answer (ii)

Q.5 ***Which of the following differential equations are exact***

(a) $\frac{dx}{x} - \frac{dy}{y} = 0$ *(b)* $3x^2y\,dx + (y + x^3)\,dy = 0$

(A) Both (a) and (b) (B)only (a) (C) only (b)

(D) None of (a) and (b)

Answer (A)

Q.6 ***The degree of the following differential equation***

$$\left(\frac{d^1y}{dx^1}\right)^3 + y\left(\frac{dy}{dx}\right)^4 = 7y \text{ is}$$

(A) 2 (B)1 (C) 3 (D) 4

Answer (C)

Q.7 ***The necessary and sufficient condition for a differential equation of first order and first degree to be exact is $Mdx + Ndy = 0$***

(A) $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ (B) $\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}$ (C) $\frac{\partial M}{\partial x} - \frac{\partial M}{\partial y}$ (D) None of these

Answer (A)

Q.8 ***Clairaut's equation is defined by***

(A) $y = px$ (B) $y = px + f(P)$ (C) $y = Px + f(c)$ (D) None of these

Answer (B)

Q.9 ***The degree of the following differential equation $\frac{d^1y}{dx^1} + 5\frac{dy}{dx} + \int y\,dx = x^3$ is***

(A) 2 (B) 3 (C) 1 (D) None of these

Answer-©

Q.10 ***if $(x^2 - 9y)dx = (ax - y^2)dy$ then $\frac{\partial M}{\partial y}$?***

(A) a (B) –a (C)-3a (D) a^2

Answer (B)

Unit- 2

Linear Differential Equation with constant coefficient, C.F. and P.I., Differential Equation of first order but not of first degree

Differential equations of first order but not of first degree

The general form of the differential equation of nth degree and first order can be written as

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \ldots + P_{n-1}\left(\frac{dy}{dx}\right) + P_n = 0$$

For the sake of convenience $\left(\frac{dy}{dx}\right)$ will be denoted by p.

$$p^n + P1p^{n-1} + P_2\, p^{n-2} + \cdots + P_{n-1}p + P_n = 0 \qquad \text{-------------1}$$

Or $F\ (x, y, p) = 0$

Where $P_1, P_2, \ldots P_n$ are function of x and y

Equation solvable for p :-

Let's suppose the left side of equation (1) can be factorized into rational factors of first degree then equation (1) can be written in this form –

$(p - Q_1)$ $(p - Q_2)$ $(p - Q_n)$ =0

Where $Q_1, Q_2 \ldots\ldots Q_n$ are function of x and y then each factor when equated to zero will give a solution. In all there will be n solution of the type $F_1(x, y, C_1) = 0$ $F_2(x, y, C_2) = 0$

$F_n(x, y, C_n) = 0$ where $C_1, C_2 \ldots\ldots C_n$ are arbitrary constant of integration. The general solution of equation (1) can be written as $F_1(x, y, C)$ $F_2(x, y, C) \ldots\ldots\ldots F_n(x, y, C) = 0$

Equation solvable for x:- If the given differential equation is solvable for x, let it be put in the form $x = F\ (y, p)$ (1) Whose differentiation with respect to y will yield

$$\frac{dx}{dy} = \frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \qquad \text{-------------2}$$

Lets the solution of Equation (2) is

$F_2(y, p, c) = 0$ -------------3

Elimination of p between the equation (1) and (3) will give the required solution of given equation .

Equation solvable for y

If the given differential Equation is solvable for y. Let it be put in the form $y = f(x, p)$……(1)

Whose differentiation with respect to x yield

$\frac{dy}{dx} = p = \phi \left[x, p, \frac{dp}{dx}\right]$ ------(2)

Lets the solution of Eq (2) is $\Psi(x, p, c) = 0$ --- (3) the elimination of p between (1) and (3) will yield a relation involving x, y *and* c and these will be the required solution.

Clairaut's Equation : Differential equation $y = px +$ f (p) is known as clairaut's equation. The general solution of clairaut's equation will be $y = cx +$ f(c)

Linear differential Equation with constant coefficient :-

Differential Equation of the form

$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = Q(x)$ ------- (1)

Where $P_1, P_2, \ldots .. P_{n-1}, P_n$ and Q are either constants or function of x is a linar differential equation of nth order.

If $P_1, P_2, \ldots .. P_{n-1}, P_n$ are constants and Q is function of x then Eq (1) are called linear differential equation with constant coefficient. Its can be written in this form

$F(D)y = Q(x)$ where $D^n = \frac{d^n}{dx^n}$

And $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$ D *is called differential operator*

Complementary function and particular Integral.

Equation $f(D)y = Q(x) =$ $-----1$

If $Q(x) = 0$ in Equation (1) then Equation $f(D)y = 0$ ___(2) is known as Homogeneous part of linear differential equation.

If $y = y_1, y = y_2, \ldots. y = y_n$ *be n linearly*

Independent solution of (2) then $y = c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots + c_n y_n$ …………..3

Will be solution of Eq. (2) where $c_1, c_2 \ldots.. c_n$ are n arbitrary constant. A relation containing n arbitrary constant is called the complete integral of a differential Equation of nth order.

Eq. (3) is called complementary function of differential Equation (1)

C.F. = $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = u(x)$

Now let $y = u(x) + v(x)$ where $v(x)$ is particular solution of equation (1)

i.e. P.I. = $v(x)$

General solution of a linear differential Equation

General solution = complementary function +Particular Integral

Y = C.F.+P.I.

Method of finding out the particular integral

Equation $f(D)y = Q(x)$ -------1

$$Q(x) \neq 0$$

$y = \frac{Q(x)}{F(D)}$ ---------2

Eq. (2) is particular Integral of Equation (1)

Case I- If $F(D)$=D

Then P.I.= $\frac{1}{D}\ Q(x) = \int Q(x)dx$

Case II- If $f(D) = D - \alpha$

Then $P.I. = y\ = \frac{1}{D-\alpha}\ \ Q(x)$

Or $(D-\alpha)y = Q(x)$

Or $\frac{dy}{dx} - \alpha y = Q(x)\ \Rightarrow y = e^{\alpha x}\int Q(x)e^{-\alpha x}dx$

$\therefore\ \frac{1}{D-\alpha}\ \ Q(x) = e^{\alpha x}\int Q(x)e^{-\alpha x}dx$

<u>Case III</u> If $F(D) = (D-\alpha_1)\ (D-\alpha_2)$____$(D-\alpha_n)$

Then P.I. = $\frac{1}{(D-\alpha_1)(D-\alpha_2)\ldots\ldots(D-\alpha_n)}$

$= A_1 e^{\alpha_1 x}\int Q(x)e^{-\alpha_1 x}dx + A_2 e^{\alpha_2 x}\int Q(x)e^{-\alpha_2 x}dx$ + --------+ $A_n e^{-\alpha_n x}\int Q(x)e^{-\alpha_n x}dx$

Case IV If $Q\ (x) = e^{ax}$ where a is any constant

Then P.I. = $\frac{1}{f(D)}.\ e^{ax} = \frac{e^{ax}}{f(a)},\quad f(a)\ \neq 0$

And $\frac{1}{f(D)}e^{ax}\ = \ \frac{1}{\phi(a)}\ \frac{x^r}{r!}\ e^{ax},\ \ \phi(a)\ \ \neq 0$

Case V- if $Q\ (x) = \sin a\ x$ where a is any constant

$\therefore$ P.I. = $\frac{1}{f(D^2)}$ sinax = $\frac{1}{f(-a^2)}$ sinax, $f(-a^{2)}$ ≠0

And $\frac{1}{D^2+a^2}$ sinax = $\frac{-x}{2a}\ cosa\ x$, $\quad f\ (-a^2) = 0$

Case VI If $Q(x) = Cosax$, where a is any constant

$$\therefore P.I. = \frac{1}{F(D^2)} Cosax = \frac{1}{F(-a^2)} Cosax, \quad f(-a^2) \neq 0$$

And $\frac{1}{D^2+a^2} Cosax = \frac{x}{2a} Sinax, \quad f(-a^2) = 0$

Case VII- If $Q(x) = x^m$ where m is any positive integer. Then P.I =

$\frac{1}{f(D)} x^m$ if $f(D) = D - \alpha$ then

P.I. = $\frac{1}{(D-\alpha)} x^m = -\frac{1}{\alpha(1-\frac{D}{\alpha})} x^m$

$= \frac{-1}{\alpha} \left(x^m + \frac{mx^{m-1}}{\alpha} + \cdots . + \frac{m!}{\alpha^m}\right)$

Case VIII- If $Q(x) = e^{ax} V$ where a is constant and V is any function of x

Then P.I. = $\frac{1}{F(D)} e^{ax} v = e^{ax} \frac{1}{F(D+a)} v$

Q.1 **Solve** $\boldsymbol{P^3 - 4xyp + 8y^2 = 0}$ **where** $\boldsymbol{P = dy/dx}$

Solution $P^3 - 4xyp + 8y^2 = 0$ ---- (1)

The given equation (1) can be written as –

$x = \frac{2y}{P} + \frac{P^2}{4y}$ ------ (2)

Differentiating Eq (2) with respect to y

$$\frac{1}{P} = \frac{2}{P} - \frac{2y}{P^2}\frac{dp}{dy} - \frac{P^2}{4y^2} + \frac{P}{2y}\frac{dp}{dy}$$

Or $\left(\frac{1}{P} - \frac{2y}{P^2}\frac{dP}{dy}\right) + \left(\frac{P}{2y}\frac{dP}{dy} - \frac{P^2}{4y^2}\right) = 0$

Or $\frac{1}{P}\left(1 - \frac{2y}{P}\frac{dP}{dy}\right) - \frac{P^2}{4y^2}\left(1 - \frac{2y}{P}\frac{dP}{dy}\right) = 0$

Or $\left(1-\frac{2y}{P}\frac{dP}{dy}\right)\left(\frac{1}{P}-\frac{P^2}{4y^2}\right)=0$ ……….(3)

Here we neglect the factor $\left(\frac{1}{P}-\frac{P^2}{4y^2}\right)$

Because its gives us singular solution therefore

$1-\frac{2y}{P}\frac{dP}{dy}=0$ or $\frac{dy}{y}-\frac{2dP}{P}=0$

On Integrating we obtain

$logy-2logP+logc=0$

Or $P^2=cy$ --------(4)

From Eq. (2) and (4)

$x=\frac{2y}{\sqrt{cy}}+\frac{cy}{4y}$ or $x-\frac{c}{4}=\frac{2\sqrt{y}}{\sqrt{c}}$

Or $\frac{(4x-c)^2}{16}=\frac{4y}{c}$

Or $c(4x-c)^2=64y$ is the required solution of given equation

Q.2 **Solve $(D^2+1)^2y=cosx\ \cosh x$ where $D=d/d\,x$**

Solution Here the auxiliary Equation is $(m^2+1)^2=0$ so the complementary function will be $(m^2+1)^2=0$

$\therefore\ m=ti,ti$

C.F. $y=(c_1+c_2x)\cos x+(c_3+c_4x)\sin x$

Particular integral will be given by

$y=\frac{1}{(D^2+1)^2}\cos x\ coshx$

I.F. $=\frac{Cos\ h\ x\ Cos\ x}{(D^2+1)^2}$

I.F. $=\frac{1}{2}\frac{(e^x+e^{-x})\cos x}{(D^2+1)^2}$

$$= \frac{1}{2}\left[\frac{(e^{x}\cos x)}{(D^2+1)^2} + \frac{(e^{-x}\cos x)}{(D^2+1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{(D+1)^2+1)^2} + \frac{e^{-x}\cos x}{\{(D-1)^2+1\}^2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{(D^2+2D+2)^2} + \frac{e^{-x}\cos x}{(D^2-2D+2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{(-1+2D+2)^2} + \frac{e^{-x}\cos x}{(-1-2D+2)^2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{(2D+1)^2}\right] + \left[\frac{e^{-x}\cos x}{(1-2D)^2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{(4D^2+1+4D)} + \frac{e^{-x}\cos x}{(4D^2-4D+1)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{-4+1+4D}\right] + \left[\frac{e^{-x}\cos x}{-4-4D+1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x}{4D-3}\right] + \left[\frac{e^{-x}\cos x}{-(4D+3)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{x}\cos x(4D+3)}{16D^2-9} - \frac{e^{-x}\cos x\,(4D-3)}{16D^2-9}\right]$$

Q.3 Solve $\frac{d^1 y}{dx^1} - \frac{2dy}{dx} + y = xe^{x}\sin x$

Sol. Here the auxiliary Equation is $m^2 - 2m + 1 = 0$

So the C.F. will be- $m = 1,\ 1$

C.F. = $y = (c_1 + c_2 x)e^{x}$

P.I. = $\frac{1}{D^2-2D+1}\{xe^{x}\sin x\}$

$= e^{x}\ \frac{1}{(D+1)^2-2(D+1)+1}\ \{x\sin x\}$

$= e^{x}\ \frac{1}{D^2}\{x\ \sin x\}$

$= e^{x}\ \frac{1}{D}\{-x\ \cos x + \int \cos x\ dx\}$

$= -e^{x}\{x\ \sin x + 2\cos x\}$

Hence the general solution of the given equation is

$$y = (c_1 + c_{2x})\, e^x - e^x(x\sin x + 2\cos x)$$

Q.4 Solve $x^2p^2 - 2xyP + 2y^2 - x^2 = 0$

Sol. $x^2p^2 - 2xyP + 2y^2 - x^2 = 0$

$$P = \frac{2xy \pm \sqrt{\{4x^2y^2 - 4x^2(2y^2 - x^2)\}}}{2x^2}$$

Or $P = \frac{y \pm \sqrt{x^2 - y^2}}{x}$

Or $\frac{dy}{dx} = \frac{y}{x} \pm \sqrt{1 - \frac{y^2}{x^2}}$

Equation (1) is Homogenous Equation therefore put

$y = vx$ so that $\frac{dy}{dx} = v + \frac{xdv}{dx}$

$$v + \frac{xdv}{dx} = v \pm \sqrt{1 - v^2}$$

$$\frac{dv}{\sqrt{1-v^2}} = \pm\frac{dx}{x}$$

On integration we obtain

$$\sin^{-1} v = \pm \log x \pm \log c$$

Or $\sin^{-1}\frac{y}{x} \pm \log ex$

Q.5 $\frac{d^3y}{dx^3} + a^2\frac{dy}{dx} = \sin ax$

Sol. Here the auxiliary Equation is $m^3 + a^2m = 0$

Or $m = 0, m = \pm ia$

$\therefore C.F. = c_1 + c_2 \cos ax + c_3 \sin ax$

P.I. $= \frac{1}{D(D^2+a^2)} \sin ax = \frac{1}{D}\left\{\frac{1}{D^2+a^2} \sin ax\right\}$

$= \frac{1}{D}\left\{\frac{-x}{2^a}\ cosax\right\} = \frac{-1}{2^a}\int x\ cosax\ dx$

$= \frac{-1}{2^a}\left[\frac{x}{a}\ sin\ ax + \frac{1}{a^2}\ cosax\right]$

$\frac{-x}{2a^2}\ sinax - \frac{1}{2a^3}\ cosax$

Hence the general solution of the given equation is

$y = C.F. + P.I. = c_1 + c_2\ cosax + c_3\ sinax - \frac{x}{2a^2}\sin ax$

Q.6 Solve $y = 2px + y^2p^2$

Sol. The given equation can be written as

$x = \frac{y}{2p} - \frac{y^2p^2}{2}$ ---------(1)

Differentiating Equation (1) with respect to y

$\frac{1}{P} = \frac{1}{2P} - \frac{-y}{2P^2}\frac{dp}{dy} - yp^2 - y^2p\frac{dp}{dy}$

$\frac{1}{P} = \frac{1}{2P} - \frac{-y}{2P^2}\frac{dp}{dy} - yp^2 - y^2p\frac{dp}{dy}$

$or\ \left(\frac{1}{2P} + yp^2\right) + \frac{y}{P}\left(\frac{1}{2P} + yp^2\right)\frac{dp}{dy} = 0$

$or\ \left(\frac{1}{2P} + yp^2\right)\left(1 + \frac{y}{P}\frac{dp}{dy}\right) = 0$ ----------(2)

Here we neglect the factor $\left(\frac{1}{2P} + yp^2\right)$ because its gives us singular solution. Therefore $1 + \frac{y}{P}\frac{dp}{dy} = 0\ or\ \frac{dy}{y} + \frac{dp}{p} = 0$

On Integration we obtain $\log y + \log P = \log c$

Or $py = c\ or\ P = \frac{c}{y}$ -------------(3)

From Equation (1) and (3)

$x = \frac{y^2}{2c} - \frac{c^2}{2}$

$y^2 = 2cx + c^3$ is the general solution of given equation.

Q.7 $(Px - y)\ (py + x) = h^2P$

Sol. Suppose $x^2 = u$ and $y^2 = v$ then

$\frac{dv}{du} = \frac{2y}{2x}\ \frac{dy}{dx} \qquad = \frac{y}{x}\ \frac{dy}{dx}$

Suppose

$\therefore\ P = \frac{x}{y}\ \frac{dv}{du} = \frac{x}{y}.P \qquad \left[P = \frac{dv}{du}\right]$

Substitute $\frac{x}{y}\ P$ in place of P in the given Equation.

$\left(\frac{x}{y}Px - y\right)\ \left(\frac{x}{y}Py + x\right)\ = h^2\frac{x}{y}P$

Or `

$(Px^2 - y^2)\ (P + 1) = h^2p$

$(Pu - v)\ (P + 1) = h^2p$

$v = Pu - \frac{h^2P}{P+1}$ which is in the form of clairaut's Equation

Therefore the required solution will be

$v = cu - \frac{h^2c}{c+1}$ where c is any arbitrary constant

$y^2\ cx^2 - \frac{h^2c}{c+1}$

Q.8 Solve $(D^4 - m^4)y = coshmx$

Sol. Here the auxiliary equation is $M^4 - m^4 = 0$

Or $(M - m)\ (M + m)\ (M^2 + m^2) = 0$

$\therefore\ M = m, -m, \pm mi$

C.F = $c_1e^{mx} + c_2e^{-mx} + c_3cosmx + c_4sinmx$

P.I. $= \frac{1}{D^4 - m^4}$ coshmx

$$= \frac{1}{(D-m)(D+m)(D^2+m^2)} \frac{e^{mx}+e^{-mx}}{2}$$

$$= \frac{1}{2(D-m)(D+m)(D^2+m^2)} e^{mx}$$

$$+\frac{1}{2} \frac{1}{(D-m)(D+m)(D^2+m^2)} e^{-mx}$$

$$= \frac{1}{2} \frac{x}{!} \frac{e^{-mx}}{(m+m)(m^2+m^2)} + \frac{1}{2} \frac{x}{!} \frac{e^{-mx}}{(-m-m)(m^2+m^2)}$$

$$= \frac{x\, e^{mx}}{8m^3} - \frac{x\, e^{-mx}}{8m^3}$$

$$= \frac{x}{4m^3}\left(\frac{e^{mx}-e^{-mx}}{2}\right) = \frac{x}{4m^3} \sinh mx$$

Therefore the general solution of given Equation

$$y = C.F. + P.I.$$

$$y = c1 e^{mx} + c_2 e^{-mx} + c_3 \cos ms + c_4 \sinh mx$$

$$+ \frac{x}{4m^3} \sinh mx$$

Q.9 **Solve** $\frac{d^3y}{dx^3} + 2\frac{d^1y}{dx^1} + \frac{dy}{dx} = e^{2x} + x^2 + x$

Solution The given Equation can be written as $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$

Where $D = \frac{d}{dx}$

Here the auxiliary Equation is $m^3 + 2m^2 + m = 0$

or $m(m+1)^2 = 0 \therefore m = 0, -1, -1$

therefore C.F $= c_1 + (c_2 + c_3 x)e^{-x}$

P.I. $= \frac{1}{D(DH)^2} (e^{2x} + x^2 + x)$

$$= \frac{1}{D(DH)^2} e^{2x} + \frac{1}{D}(1+D)^{-2} (x^2 + x)$$

$$= \frac{1}{2(2+1)^2} e^{2x} + \frac{1}{D}(1 - 2D + 3D^2 - 4D^3 +) (x^2 + x)$$

$$= \frac{1}{18}e^{2x} + \left(\frac{1}{D} - 2 + 3D - 4D^2 + \dots\right)(x^2 + x)$$

$$= \frac{1}{18}e^{2x} + \left[\frac{x^3}{3} + \frac{x^2}{2} - 2x^2 - 2x + 6x + 3 - 8\right]$$

$$= \frac{1}{18}e^{2x} + \frac{x^3}{3} - \frac{3}{2}x^2 + 4x - 5$$

Therefore the general solution of given Equation

y=C.F. +P.I.

$$= c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18}e^{2x} + \frac{x^3}{3} - \frac{3}{2}x^2 + 4x$$

Q.10 **Solve** $(D^2 + a^2)y = \sec ax$

Solution Here the auxiliary Equation is $m^2 + a^2 = 0$

Or $m = \pm ia$

$\therefore C.F = c_1 cosax + c_2\ sinax$

$$P.I. = \frac{1}{D^2 + a^2}\ secax = \frac{1}{(D+ia)(D-ia)} secax$$

$$= \frac{1}{2ia}\ \left(\frac{1}{D-ia} - \frac{1}{D+ia}\right)\ Secax$$

$$= \frac{1}{2ia}\ \left(\frac{1}{D-ia} Secax - \frac{1}{D+ia}\ Secax\right)$$

$$\frac{1}{2ia}\ \{e^{iax} \int secax\ e^{-iax}\ dx - e^{-iax} \int secax\ e^{iax}\ dx\}$$

$$= \frac{1}{2ia}\{e^{iax} \int secax\ (cosax - i\ sinax)dx$$

$$-e^{-iax}\int secax\ (cosax - i\ sinax)dx\}$$

$$= \frac{1}{2ia}[e^{iax}\ \{\int dx - i\int tanax\ dx\}$$

$$-e^{-iax}\int dx - i\int tanax\ dx]$$

$$= \frac{1}{2ia}\{e^{iax}\ (x + (i/a) \log cosax) - e^{-iax}\ (x - (i/a) \log cosax)\}$$

$$= \frac{1}{2ia}\left\{x\left(e^{iax} - e^{-iax}\right) + \frac{i}{a}\log\ cosax\left(e^{iax} + e^{-iax}\right)\right\}$$

$$P.I. = \frac{1}{2ia}\left\{2ix\ sinax\ + \frac{2i}{a} cosax \log cosax\right.$$

$$P.I. = \frac{x}{a}\left\{\ sinax\ + \frac{1}{a^2} cosax \log cosax\right.$$

Therefore the general solution of given Equation

$\mathbf{y} = \text{C.F.} + \text{P.I}$

$$= c_1\ cosax + c_2 sinax + \frac{x}{a}\ sinax + \frac{1}{a^2}\ cosax \log cosax$$

Q.11 **Solve :** $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = \mathbf{1 + x^2}$

Solution The given equation can be written as

$(D^3 - D^2 - 6D)\ y = 1 + x^2$

Here the auxiliary equation is $m^3 - m^2 - 6m = 0$

Or $m(m+2)(m-3) = 0$

$\therefore 0, -2, 3 = m$

Therefore C.F. $= c_1 + c_2\ e^{-2x} + c_3 e^{3x}$

$$P.I. = \frac{1}{D\ (D+2)\ (D-3)}\ (1 + x^2)$$

$$\frac{1}{D\ (D^2-D-6)}\ (1 + x^2) = \frac{-1}{6D}\left(1 + \frac{D-D^2}{6}\right)^{-1}\ (1 + x^2)$$

$$= \frac{-1}{6D}\left(1 - \frac{D-D^2}{6} + \frac{(D-D^2)^2}{36} + \cdots\right)(1 + x^2)$$

$$= \frac{-1}{6}\left(\frac{1}{D}\ \ \frac{-1}{6} + \frac{D}{6} + \frac{D}{36} - \frac{D^2}{18} + \cdots\right)(1 + x^2)$$

$$= \frac{-1}{6}\left(\frac{1}{D}\ \ \frac{-1}{6} + \frac{7}{36}D - \frac{D^2}{18} + \cdots\right)(1 + x^2)$$

$$= \frac{-1}{6}\left(x\ + \frac{x^3}{3} - \frac{1}{6} - \frac{x^2}{6} + \frac{7}{18}x\,\frac{-1}{9}\right)$$

$= \frac{-25}{108}x + \frac{1}{36}x^2 - \frac{1}{18}x^3 + constant$

Therefore the general solution of given Equation

$y = C.F. + P.I.$

$= c_1 + c_2\, e^{-2x} + c_3\, e^{3x} - \frac{25}{108}x + \frac{1}{36}x^2 - \frac{1}{18}x^3$

Q.12 **Solve $P^2 + 2py\, cotx = y^2$**

Solution The given Equation can be written as

$P^2 + 2py\, cotx - y^2 = 0$

$P = \frac{-2y\, cotx \pm \sqrt{4y^2\, cot^2 x + 4y^2}}{2}$

Or $P = -y\, cotx \pm ycosax$ ----------(1)

Taking positive sign in equation (1)

$P = -ycotx + ycosecx$

$or\ \frac{dy}{dx} = y\left(\frac{1-\cos x}{\sin x}\right) = y\left[\frac{1-cos^2 x}{sinx\,(1+\cos x)}\right]$

Or $\frac{dy}{dx} = y\left(\frac{sinx}{1+\cos x}\right)$ $or\ \frac{dy}{y} = \left(\frac{sinx}{1+\cos x}\right)dx$

On Integration logy $= -\log(1 + cosx) + logc$

$\therefore\ y = \frac{c}{1+cosx}$ ------(2)

Taking negative sign in Equation (1)

$P = -ycotx - y\, cosecx$

$or\ \frac{dy}{dx} = -y\left(\frac{1+\cos x}{\sin x}\right) = -y\left[\frac{\sin x}{(1-\cos x)}\right]$

$or\ \frac{dy}{y} = -\left(\frac{sinx}{1-cosx}\right)dx$

On integration logy $- \log(1 - cosx) + logc$

$\therefore\ y = \frac{c}{1-\cos x}$ $\qquad ------(3)$

The general solution of given equation from equation

(2) and (3) $\left(y - \frac{c}{1+\cos x}\right)\left(y - \frac{c}{1-\cos x}\right) = 0$

Q.13 **Solve $x^2 + p^2 x = yp$**

Solution The given Equation can be written as

$y = \frac{x^2}{P} + px$ $\qquad .....(1)$

Differentiating Equation (1) with respect to x

$P = \frac{dy}{dx} = \frac{2x}{P} - \frac{x^2}{P^2}\ \frac{dp}{dx} + P + x\ \frac{dp}{dx}$

$or\ \left(x - \frac{x^2}{p^2}\right)\ \frac{dp}{dx} + \frac{2x}{P} = 0$

Or $(P^2 - x)\ \frac{dp}{dx} + 2P = 0$ or $\frac{dx}{dP} - \frac{x}{2P} = \frac{-1}{2}\ P$ ---- (2)

Is a linear differential Equation

$\therefore\ I.F. = \overline{e}^{\int \frac{1}{2P}\ dP} = e^{-\frac{1}{2}\log P} = \frac{1}{\sqrt{P}}$

Therefore the required solution of Equation (2) will be

$x.\frac{1}{\sqrt{P}} = \int \frac{-1}{2}\ P.\frac{1}{\sqrt{P}}\ dp + C$ or $\frac{x}{\sqrt{P}} = \frac{-1}{3}\ P^{3/2} + C$

Or $x = c\sqrt{P} - \frac{1}{3}P^2$ $\qquad ------(3)$

From Equation (3) and (1)

$y = \frac{1}{P}\ \left(C\sqrt{P} - \frac{1}{3}P^2\right)^2 + P\ \left(C\sqrt{P} - \frac{1}{3}P^2\right)$ $\qquad$ ----- (4)

Equation (3) and (4) both together gives the required solution of given Equation.

Q. 14 **Solve $y - 2xP + ayp^2 = 0$**

Solution : $y - 2xp + ayP^2 = 0$ ------(1)

Put $y^2 = v$ so that $2y\frac{dy}{dx} = \frac{dv}{dx}$ $in\ Eq\ (1)$ suppose $\left[\frac{dv}{dx} = P\right]$ or $2yP = P$

$y - 2x\left(\frac{P}{2y}\right) + ay\left(\frac{P^2}{4y^2}\right) = 0$

Or $4y^2 - 4xP + aP^2 = 0$

Or $4v - 4xP + aP^2 = 0$

Or $v = xp - \frac{1}{4}ap^2$ is in the form of clairaut's form

$\therefore v = xc - \frac{1}{4}ac^2$

$y^2 = cx - \frac{1}{4}ac^2$

Q.15 $\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} = x^2 + 3e^{2x} + 4\sin x$

Solution The given Equation can be written as

$(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x} + 4\sin x$

Here the Auxiliary Equation is $m^4 + 2m^3 - 3m^2 = 0$

Or $m^2\ (m^2 + 2m - 3) = 0$ or $m^2\ (m-1)(m+3) = 0$

$\therefore m = 0, 0, 1, -3$

$\therefore C.F. = (c_1 + c_2x) + c_3e^x + c_4e^{-3x}$

P.I. $= \frac{1}{D^2\ (D-1)\ (D+3)}\ (x^2 + 3e^{2x} + 4sinx)$

$= \frac{1}{D^2\ (D^2+2D-3)}x^2 + \left(\frac{1}{D^4+2D^3-3D^2}\right)(3e^{2x} + 4sinx)$

$= \frac{-1}{3D^2}\left(1 - \frac{2D+D^2}{3}\right)^{-1}x^2 + \frac{3e^{2x}}{2^4+2.2^3-3.2^2}\Bigg) + 4.\frac{1}{(-1)^2+2D\ (-1)-3(-1)}\sin x$

$= \frac{-1}{3D^2}\left(1 + \frac{2D+D^2}{3} + \frac{(2D+D^2)^2}{9} + \cdots\right)x^2 + \frac{3e^{2x}}{20} + 2\ \frac{(2+D)}{4-D^2}\ sinx$

$= \frac{-1}{3D^2}\left(x^2 + \frac{4x}{3} + \frac{2}{3} + \frac{8}{9}\right) + \frac{3}{20}\ e^{2x} + \frac{2(2+D)sinx}{4-(-1)}$

$= \frac{-1}{3}\left(\frac{x^4}{12} + \frac{2}{9}x^3 + \frac{14}{9} + \frac{x^2}{2}\right) + \frac{3}{20}\ e^{2x} + \frac{2}{5}\ (2\sin x + \cos x)$

Therefore the general solution of given Equation is $y = C.F. + P.I.$

$$= c_1 + c_2 x + c_3 e^x + c_4 e^{-3x} + \frac{3}{20} e^{2x}$$

$$+ \frac{2}{5} (2\sin x + \cos x) - \frac{1}{3}\left(\frac{x^4}{12} + \frac{2}{9}x^3 + \frac{7}{9}x^2\right)$$

Q. 16. **Solve $\frac{d^1 y}{dx^1} - 3\frac{dy}{dx} + 2y = e^x$**

Solution The given equation can be written as

$(D^2 - 3D + 2)\, y = e^x$

Here the auxiliary Equation is $m^2 - 3m + 2 = 0$

Or $(m-2)(m-1) = 0$ or $m = 2{,}1$

C.F. $= c_1 e^x + c_2 e^{2x}$

P.I. $\frac{1}{(D-1)(D-2)} e^x = \frac{1}{D-1}\left\{\frac{1}{(D-2)} e^x\right\}$

$= \frac{1}{D-1}\left\{\frac{1}{(1-2)} e^x\right\} = \frac{1}{(D-1)} e^x$

$= -\left[e^x \int e^x\, e^{-x}\, dx\right] = -x e^x$

Therefore the general solution of given Equation is

$y = C.F. + P.I.$

$y = c_1 e^x + c_2 e^{2x} - x e^x$

Q. 17 **Solve $(D^2 - 1)y = \cosh x \cos x$**

Solution The given Equation can be written as

$(D^2 - 1)y = \frac{1}{2}(e^x + e^{-x})\cos x$

Here the auxiliary Equation is $m^2 - 1 = 0$

Or $(m+1)(m-1) = 0$ or $= m = 1, -1$

$\therefore\ C.F. = c_1 e^x + c_2 e^{-x}$

$P.I. = \frac{1}{(D+1)(D-1)}\left\{\frac{1}{2}(e^x + e^{-x})\cos x\right.$

$= \frac{1}{2}\frac{1}{(D+1)(D-1)} e^x \cos x + \frac{1}{2}\frac{1}{(D+1)(D-1)} e^x \cos x$

$$= \frac{1}{2}\ e^{x} \frac{1}{D^2+2D}\ \cos x + \frac{1}{2}\ e^{-x}\ \frac{1}{D^2-2D}\ \cos x$$

$$= \frac{1}{2}\ e^{x} \frac{1}{-1+2D}\ \cos x + \frac{1}{2}\ e^{-x}\ \frac{1}{-1-2D}\ \cos x$$

$$= \frac{1}{2}\ e^{x} \frac{2D+1}{4D^2-1}\ \cos x - \frac{1}{2}\ e^{-x}\ \frac{2D-1}{4D^2-1}\ \cos x$$

$$= \frac{1}{2}\ e^{x} \frac{2D+1}{-4-1}\ \cos x - \frac{1}{2}\ e^{-x}\ \frac{2D-1}{-4-1}\ \cos x$$

$$= \frac{-1}{10}\ e^{x}(-2sinx + \cos x)\frac{+1}{10}\ e^{-x}(-2sinx - \cos x)$$

$$= \frac{2}{5}\ sinx\ \left(\frac{e^{x}-e^{-x}}{2}\right) - \frac{cosx}{5}\ \left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$\frac{2}{5}\ sinx\ sinhx\ - \frac{1}{5}\ cosx\ coshx$$

Therefore the general solution of given equation is $y = C.F. + P.I.$

$$c_1e^{x} + c_2e^{-x} + \frac{2}{5}\ sinx\ sinhx - \frac{1}{5}\ cosx\ coshx$$

Q.18 $(D^2 + a^2)y = tanax$

Solution Here the auxiliary Equation is $m^2 + a^2 = 0$ or $m = \pm ai$

C.F. $= c_1 cosax + c_2 sinax$

P.I. $= \frac{1}{D^2+a^2}\ tanax = \frac{1}{(D+ia)(D-ia)}\ tanax$

$$= \frac{1}{2ia}\ \frac{1}{(D-ia)} - \frac{1}{(D+ia)}\ tanax$$

$$= \frac{1}{2ia}\ \left[e^{iax}\int e^{-iax}tanax\ dx - e^{-iax}\int e^{iax}tanax\ dx\right]$$

$$= \frac{1}{2ia}\ \left[\left\{e^{iax}\int\left(sinax - i\ \frac{sin^2 ax}{cosax}\right)dx\right\}\right.$$

$$\left. - \left\{e^{-iax}\int\left(sinax + i\ \frac{sin^2 ax}{cosax}\right)dx\right\}\right]$$

$$= \frac{1}{2ia}\ \left[e^{iax}\int sinax\ dx - ie^{iax}\int\left(\frac{1-cos^2 ax}{cosax}\right)dx\right.$$

$$\left. -e^{-iax}\int\left(sinaxdx - ie^{iax}\ \int\frac{1-cos^2 ax}{cosax}\right)dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left(\frac{-cosax}{a}\right) - ie^{iax}\left\{\frac{1}{a}\log tan\left(\frac{\Pi}{4}+\frac{ax}{2}\right) - \frac{1}{a\ sinax}\right\}\right.$$

$$\left. -e^{-iax}\left(\frac{-cosax}{a}\right) - ie^{-iax}\left\{\frac{1}{a}\log tan\left(\frac{\Pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}sinax\right\}\right]$$

$$= \frac{1}{2ia}\left[\frac{-1}{a}e^{iax}\left\{(cosax - i\ sinax) + i\ logtan\left(\frac{\Pi}{4}+\frac{ax}{2}\right)\right\}\right.$$

$$\left. +\frac{1}{a}e^{-iax}(cosax - i\ sinax) - i\ logtan\left(\frac{\Pi}{4}+\frac{ax}{2}\right)\right\}\right]$$

$$= \frac{1}{2ia}\left[\frac{-1}{a}\left\{1 + ie^{iax}logtan\left(\frac{\Pi}{4}+\frac{ax}{2}\right)\right\}\right.$$

$$\left. +\frac{1}{a}\left\{1 - ie^{-iax}logtan\left(\frac{\Pi}{4}+\frac{ax}{2}\right)\right\}\right]$$

$$= \frac{1}{2ia}\left[\frac{-i}{a}\left(e^{iax} + e^{-iax}\log tan\left(\frac{\Pi}{4}+\frac{ax}{2}\right)\right)\right]$$

$$= +\frac{-1}{a^2}\cos ax\, logtan\left(\frac{\Pi}{4}+\frac{ax}{2}\right)$$

Therefore the required general solution is

Y=C.F +P.I.

$$= c_1cosax + c_2sinax - \frac{1}{a^2}cosax\log tan\left(\frac{\Pi}{4}+\frac{ax}{2}\right)$$

Q.19 $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + y = ax^2 + be^{-x}sin2x$

Solution In symbolic manner above equation can be written as $(D^4 + D^2 + 1)\ y = ax^2 + be^{-x}sin2x$

Here the auxiliary Equation is $m^4 + m^2 + 1 = 0$ or $(m^2+1)^2 - m^2 = 0$ or $(m^2 + 1 + m)\ (m^2 + 1 - m) = 0$

$m^2 + m + 1 = 0$, gives $m = \frac{-1 \pm \sqrt{3}\ i}{2}$

And $(m^2 - m + 1) = 0$, gives $m = \frac{1 \pm \sqrt{3}\ i}{2}$

Hence the complementary function will be

$$y = e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2sin\frac{\sqrt{3}}{2}x\right]$$

$= e^{-x/2}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 sin\frac{\sqrt{3}}{2}x\right]$

P.I. $=\frac{1}{D^4+D^2+1}\ (ax^2 + be^{-x} sin2x)$

$= a(\ 1 + D^2 + D^4)^{-1}x^2 + be^{-x}\frac{1}{(D-1)^4+(D-1)^2+1}\ sin2x$

$= a(\ 1 - D^2 - \ldots)x^2 + be^{-x}\frac{1}{D^4 - 4D^3+7D^2-6D+3}\ sin2x$

$= ax^2 - 2a + be^{-x}\frac{1}{(-4)^2 - 4D(-4)+7(-4)-6D+3}\ sin2x$

$ax^2 - 2a + be^{-x}\frac{10D+9}{(100D^2-81)}\ sin2x$

$ax^2 - 2a - \frac{b}{481}e^{-x}\ (20\cos 2x + 9\ sin2x$

Therefore the complete solution will be

$y = C.F. + P.I.$

$= e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 sin\frac{\sqrt{3}}{2}x\right]$

$+ e^{x/2}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 sin\frac{\sqrt{3}}{2}x\right]$

$ax^2 - 2a - \frac{b}{481}e^{-x}(\ 20cos2x + 9\sin 2x)$

Q. 20 $\frac{d^2y}{dx^2} + \frac{dy}{dx} = xcosx$

Solution *Here the Auxiliary Equation is* $m^2 + m = 0$

Or $m\ (m+1) = 0\ \therefore m = 0, -1$

C.F. $= c_1 + c_2\ e^{-x}$

P.I. $= \frac{1}{D(D+1)}\ x\ cosx$

$= x\frac{1}{D^2+D}Cos\ x + \left[\frac{d}{dD}\left(\frac{1}{D(D+1)}\right)\right]\cos x$

$= x\frac{1}{-1+D}Cos\ x + \left[\frac{d}{dD}\left(\frac{1}{D} - \frac{-1}{D+1}\right)\right]\cos x$

$= x \frac{D+1}{(D-1)(D+1)} cosx + \left[\frac{-1}{D^2} + \frac{1}{D^2+2D+1}\right] \cos x$

$= \frac{-x}{2}(D+1)cosx + \left[\cos x + \frac{1}{2D}\cos x\right]$

$= \frac{-x}{2}(-sinx + cosx) + \cos x + \frac{1}{2}\sin x$

$\frac{x}{2}(sinx + cosx) + \cos x + \frac{1}{2}\sin x$

Therefore the complete solution will be

$y = c_1 + c_2 e^{-x} + \frac{x}{2}\ (sinx - cosx) + \cos x + \frac{1}{2}\sin x$

Q.21 ***Solve*** $(D^2 - 3D + 2)y = e^{2x}\sin x$

Solution *Here the auxiliary Equation is* $m^2 + 3m + 2 = 0$

Or $(m+1)(m+2) = 0\ \therefore m = ^{-1,-2}$

$C.F. = c_1\, e^{-x} + c_2\, e^{-2x}$

$P.I. = \frac{1}{(D+1)(D+2)} e^{2x}\ sinx$

$= e^{2x} \frac{1}{(D+2+1)(D+2+2)} \sin x$

$= e^{2x} \frac{1}{D^2+7D+12} \sin x = e^{2x} \frac{1}{-1+7D+12}\ sinx$

$= e^{2x} \frac{7D-11}{49D^2+121} \sin x = e^{2x} \frac{7D-11}{49(-1)-121}\ sinx$

$\frac{-1}{170}\ e^{2x}[7\cos x - 11\sin x]$

Therefore the complete solution will be

$y = C.F. + P.I.$

$c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{170}\ e^{-2x}\ [7cosx + 11\sin x]$

Q.22 **Find the general solution of the following equation.** $y = px + a/p$

Solution *The given Equation is* $y = px + \frac{a}{p}$ ---------(1)

Differentiating Equation (1) with respect to x

$$P = \frac{dy}{dx} = P + x\frac{dp}{dx} - \frac{a}{P^2}\frac{dp}{dx}$$

Or $\left(x - \frac{a}{P^2}\right)\frac{dp}{dx} = 0 \;\therefore\; \frac{dp}{dx} = 0$ ---------(2)

Or $x - \frac{a}{P^2} = 0$ -----------(3)

From Equation (2) $P = c\ (constant)$ ------------(4)

From Equation (1) and (4) $y = cx + \frac{a}{c}$ *is the general solution of given Equation.*

Q. 23 ***Solve*** $(D^2 + 6D + 9)y = 2e^{-3x}$

Solution Here the Auxiliary Equation is $m^2 + 6m + 9 = 0$

Or $(m+3)^2 = 0$ *or* $m = -3, -3$

$\therefore C.F. = (C_1 + C_2x)\,e^{-3x}$

P.I. $= \frac{1}{(D+3)^2} 2\,e^{-3x}$ [here f(a) =0]

$= 2.\frac{x^2}{2!}\,e^{-3x}$

P.I. $= x^2e^{-3x}$

Therefore the complete solution will be

$y = C.F. + P.I.$

$= (C_1 + C_2x)\,e^{-3x} + x^2e^{-3x}$

Q. 24 ***Solve*** $(D-1)^2\,(D^2+1)^2y = sin^2\frac{1}{2}x + e^x$

Solution *Here the auxiliary Equation is*

$(m-1)^2\,(m^2+1)^2 = 0 \;\therefore m = 1, 1, \pm i,\ \pm i$

C.F. $= (C_1 + C_2x)e^x + (C_3 + C_4x)\cos x + CC_5 + C_6x)\sin x$

P.I. $= \frac{1}{(D-1)^2(D^2+1)^2}\left(sin^2\frac{1}{2}x + e^x\right)$

$\frac{1}{(D-1)^2(D^2+1)^2}\ e^x + \frac{1}{(D-1)^2(D^2+1)^2}\ \frac{1-\cos x}{2}$

$\frac{x^2}{2!}\ \frac{e^x}{(1+1)^2} + \frac{1}{2}\ \frac{1}{(D-1)^2(D^2+1)^2}\ e^{ax}$

$-\frac{1}{2}\ \frac{1}{(D-1)^2(D^2+1)^2}\ cosx$

$= \frac{1}{8}x^2e^x + \frac{1}{2}\ \frac{e^{0x}}{(D-1)^2(D+1)^2}\frac{-1}{2}\ \frac{1}{(D^2-2D+1)\ (D^2+1)^2}\cos x$

$= \frac{1}{8}x^2e^x + \frac{1}{2} - \frac{1}{2}\ \frac{-1}{2}\ \frac{1}{(D^2-2D+1)\ (D^2+1)^2}\cos x$

$= \frac{1}{8}x^2e^x + \frac{1}{2} + \frac{1}{4}\ \frac{1}{(D^2+1)^2}\ \left(\frac{1}{D}\cos x\right)$

$\frac{1}{8}x^2e^x + \frac{1}{2} + \frac{1}{4}\ \frac{1}{(D^2+1)^2}\sin x$

Now $\frac{1}{(D^2+1)^2}\sin x = I.P\ of\ \frac{1}{(D^2+1)^2}\ e^{ix}$

But $\frac{1}{(D^2+1)^2}\ e^{ix} = \frac{1}{(D+i)^2(D-i)^2}\ e^{ix}$

$= \frac{x^2}{2!}\ \frac{e^{ix}}{(i+i)^2} = \frac{-1}{8}x^2\ (Cos\ x + i\sin x$

$\therefore\ \frac{1}{(D^2+1)^2}\sin x = -\ \frac{1}{8}x^2 sinx$

$\therefore\ P.I. = \frac{1}{8}x^2e^x\frac{1}{2} - \frac{1}{32}x^2\sin x$

Therefore the complete solution will be

$\boldsymbol{y = C.F. + P.I}$

$(C_1 + C_2x)e^x + (C_3 + C_4x)\cos x + (C_5 + C_6x)\sin x$

$+ \frac{1}{8}x^2e^x + \frac{1}{2} - \frac{1}{32}x^2\sin x$

Multiple Choice Questions

Q.1 ***The differential operator*** $D^m D^n$ *is equal to*

(A) D^{m-n} (B) D^{m+n} (c) $D^m + D^n$ (D) $D^{m/n}$

Answer (B)

Q.2 ***The value of*** $(D^3 - 3D^2 + 2D + 1)\, x^3 = ?$

(A) $6 - 18x + 6x^2 + x^3$ (B) $6 - 18x + 6x^3$

(C) $6 + 18x - 6x^2 + x^3$ (D) *None of these*

Answer (A)

Q.3 ***The value of*** $\frac{1}{D^2}\,(x^2) = ?$

(A) $\frac{x^3}{12}$ (B) $12x^4$ (C) $\frac{x^4}{12}$ (D) *None of these*

Answer (C)

Q.4 ***The value of*** $\frac{1}{D^2+a^2}\,\cos ax = ?$

(A) $\frac{x}{2a}\cos ax$ (B) $\frac{-x}{2a}\sin ax$ (C) $\frac{x}{a}\sin ax$ (D) $\frac{x}{2a}\sin ax$

Answer (D)

Q.5 *For the differential equation* $\frac{d^3y}{dx^3} + a^2\frac{dy}{dx} = \sin ax$ *here C.F. =?*

(A) $c_1 + c_2\cos ax + c_3\sin ax$ (B) $c_1 + c_2\sin ax + c_3\sin(-ax)$

(C) $c_1 + c_2\cos ax$ (D) *None of these*

Answer (A)

Q.6 ***For the differential Equation*** $(D^2 + a^2)y = \tan ax$ ***here C.F. =?***

(A) $c_1 e^{ax} +)\, c_2 e^{-ax}$ (B) $)\, c_1\cos ax + c_2\sin ax$

(C) $c_1\cos ax - c_2\sin ax$ (D) *None of these*

Answer (B)

Q.7 *The value of* $\frac{1}{D^1+a^1}\ sinax = ?$

(A) $\frac{-x}{2a}\ cosax$ (B) $\frac{x}{2a}\ cosax$ (C) $\frac{-x}{a}\ cosax$ (D) $\frac{x}{a}\ cosax$

Answer (A)

Q.8 *The value of-* $-\frac{1}{D^1}\ (cosax) = ?$

(A) $-cosx$ (B) $sinx$ (C) $-sinx$ (D) $cosx$

Answer (D)

Q.9 ***For the differential equation*** $(D^2+6D+9)y = 0$ ***here C.F. =?***

(A) c_1e^{-3x} (B) $c_1e^{3x}+c_2e^{-3x}$ (C) $(c_1+c_2x)e^{-3x}$ (D) *None of these*

Answer (C)

Q.10 *The Value of* $\frac{1}{(D-1)^1\ (D^1+1)^1}\ e^x$ ***is equal to***

(A) $\frac{1}{4}x^2e^x$ (B) $\frac{1}{8}x^3e^x$ (C) $\frac{-1}{8}x^2e^x$ (A) $\frac{1}{8}x^2e^x$

Answer (D)

Unit-3

Homogeneous linear differential Equation Simultaneous differential Equation, Exact linear differential Equation

Homogeneous L.D.E.

A differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x)$$

Where $a_1, a_2, \ldots a_n$ are constants and $Q(x)$ is any function of x is known as Homogeneous linear differential Equation.

Simultaneous Differential Equation :-

A simultaneous differential equation is one of mathematical equation for an indefinite function of one or more than one variable that relate the value of the function. Differentiation of an equation in various orders. Symbolically represented as

$$f_1(\boldsymbol{D})\boldsymbol{x} + f_2(\boldsymbol{D})\boldsymbol{y} = \boldsymbol{\phi_1}, (\boldsymbol{t})$$

$g_1(\boldsymbol{D})\boldsymbol{x} + g_2(\boldsymbol{D})\boldsymbol{y} = \boldsymbol{\phi_2}, (\boldsymbol{t})$ **where** $f_1, f_2, g_1\ g_2$ are polynormials in D and $\phi_1(\boldsymbol{t}), \phi_2(\boldsymbol{t})$ *are function of independent variable* $\boldsymbol{t}$.

Q.1 **Solve** $\boldsymbol{x^4}\frac{d^3y}{dx^3} + \boldsymbol{2x^3}\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 1$

Solution The given Equation is not Homogeneous linear differential equation. But in can be reduce in Homogeneous form by dividing it x.

$\therefore\ \boldsymbol{x^3}\frac{d^3y}{dx^3} + \boldsymbol{2x^2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{1}{x}$ ------- (i)

Substituting $z = log_e x \ \Rightarrow x = e^z$

Therefore Eq. (1) can be written as

$\{D(D-1)(D-2) + 2D(D-1) - D + 1\}y = e^{-z}$

Or $(D-1)^2\ (D+1)y = e^{-z}$ where $D = d/dz$

Here the auxiliary equation is $(m-1)^2\ (m+1) = 0$ or $m = 1, 1, -1$

C.F. $=(c_1 + c_2 z)e^z + c_3 e^{-z}$

$= \quad (c_1 + c_2 logx)\ x + c_3 x^{-1}$

P.I. $=\frac{1}{(D-1)^2(D+1)} e^{-z} = \frac{1}{4}\ e^{-z}\ \frac{1}{(D-1+1)}$ (1)

$\frac{1}{4}\ ze^{-z} = \frac{1}{4} x^{-1}\ logx$

Therefore the general solution will be $y = (c_1 + c_2 logx)\ x + c_3 x^{-1} + \frac{1}{4} x^{-1}\ logx$

Q.2 **Solve** $(2x^2 + 3x)\frac{d^1 y}{dx^1} + (6x + 3)\frac{dy}{dx} + 2y = (x+1)e^x$

Solution Here $P_0 = 2x^2 + 3x$ $P_1 = 6x + 3$ $P_2 = 2$ $Q = (x+1)e^x$

Now $P_2 - P_1' + P_0'' = 2 - 6 + 4 = 0$

Therefore the given differential equation is exact whose first integral will be

$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int (x+1)e^x\,dx + c_1$

Or $(2x^2 + 3x)\ \frac{dy}{dx} + \{(6x+3) - (4x + 3y)\} = xe^x + c_1$

Or $(2x^2 + 3x)\ \frac{dy}{dx} + 2xy = xe^x + c_1$ ---------- (1)

Which is not exact differential Equation but it is first order linear Equation therefore it can be written as

$\frac{dy}{dx} + \frac{2}{2x+3}\ y = \frac{e^x}{2x+3} + \frac{c_1}{x(2x+3)}$ ------(2)

Here I.F. exp.$\left[\int \frac{2}{2x+3}\ dx\right] = exp.\ [\log(2x+3)] = 2x + 3$

Therefore the required solution will be

$y.(2x+3) = \int \left(\frac{e^x}{2x+3} + \frac{c_1}{x(2x+3)}\right)(2x+3)dx + c_2$

Or $y.(2x+3) = \int \left(e^x + \frac{c_1}{x}\right)dx + c_2$

$\therefore\ y\,(2x+3) = e^x + c_1 \log x + c_2$

Q.3 Solve $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

Solution Using θ as operator we can written the given equation as

$\theta y = e^x$ where $\theta = \frac{xd}{dx}$

Or $(\theta^2 + 3\theta + 2)y = e^x$

Here the auxiliary Equation is

$m^2 + 3m + 2 - 0$

Or $(m+1)\,(m+2) = 0$

Or $m = -1, -2$

C.F. $= c_1x^{-1} + c_2x^{-2}$

Again P.I. $= \frac{1}{\theta^2+3\theta+2}\,e^x\ \frac{1}{(\theta+1)(\theta+2)}\,e^x$

$= \left[\frac{1}{\theta+1} - \frac{1}{\theta+2}\right]e^x\ \frac{1}{\theta+1}e^x - \frac{1}{\theta+2}\,e^x$

$x^{-1}\int x^{1-1}e^x dx - \ x^{-2}\int x^{2-1}e^x dx$

$x^{-1}e^x - x^{-2}\{xe^x - \int e^x\, dx\}$

$x^{-1}e^x - x^{-1}e^x + x^{-2}\, e^x = x^{-2}\, e^x$

Therefore the general solution will be

$y = c_1x^{-1} + c_2x^{-2} + x^{-2}\, e^x$

Q.4 Solve $t\,\frac{dx}{dt} + y = 0$

$t\,\frac{dy}{dt} + x = 0$

Solution The given equation is $t\,\frac{dx}{dt} + y = 0$ -----(1)

$t\,\frac{dY}{dt} + x = 0$ -------------------------------(2)

Equation (1) differentiating with respect to t

$t\ \frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$---------------------------(3)

Substitute the value of $\frac{dy}{dt}$ from Equation (2) in Equation (3)

$t^2\ \frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$ ------------------------(4)

Now put $z = log_e t \ \Rightarrow t = \ e^z\ in\ Equation\ (4)$

$\{D(D-1) + D - 1\}x = 0$ where $D = \frac{d}{dz}$ ----------(5)

Here the auxiliary Equation will be

$m^2 - 1 = 0$ or $m = \pm 1$

$\therefore\ x = c_1 e^z + c_2 e^{-z} = \ c_1 t + c_2 t^{-1}$ ---------------(6)

Again from Equation (6)

$\frac{dx}{dt} = c_1 - c_2\ t^{-2}$ -----------------(7)

From Equation (7) and (1)

$t\ (c_1 - c_2 t^{-2}) + y = 0$

$\therefore\ y = -c_1 t + c_2\ t^{-1}$ -----------------------------(8)

Equation (6) and (7) both together give the general solution of given equation

Q.5 **Solve** $\frac{d^3y}{dx^3} + \cos x\frac{d^2y}{dx^2} - 2\sin x\frac{dy}{dx} - y\cos x = \sin 2\ x$

Solution Here $P_0 = 1\ P_1 = \cos x\ \ P_2 = -2\sin x\ \ P_3 = -\cos x\ \ and\ Q = \sin 2x$

Now $P_3 - p_2' - p_1'' - p_0''' = -cosx + 2cosx - cosx - 0 = 0$

Therefore the given equation is exact whose first integral will be

or $P_0\frac{d^2y}{dx^2} + (P_1 - P_0')\frac{dy}{dx} + (p_2' - p_1' - p_0'')y$

$= \int \sin 2x\ dx + c_1$

Or $\frac{d^2y}{dx^2}+(\cos x-0)\frac{dy}{dx}+(-2\sin x+\sin x+0)y \;=\frac{-\cos 2x}{2}+c_1$

Or $\frac{d^2y}{dx^2}+\cos x\frac{dy}{dx}-sinx.\,y=\frac{-1}{2}\cos 2x+c_1$ -------------(1) Again in Equation (1)

$P_0=1\ P_1=\cos x\ P_2=-\sin x$ and $\theta=c_1-\frac{1}{2}\cos 2x$

Now

$$P_2-P_1'+P_0''=-sinx+sinx+0=0$$

Therefore the Equation (1) is also a exact equation whose first integral will be

$P_0\frac{dy}{dx}+(P_1-P_0')y=\int\{c_1-\frac{1}{2}\cos 2x\}\,dx+c_2$

Or $\frac{dy}{dx}+\cos x.\,y=c_1x-\frac{1}{4}\sin 2x+c_2$ ------------------(2)

Equation (2) is not exact equation but it is linear Equation of first order.

Here I.F. $=e^{\int\cos x\,dx}=e^{\sin x}$

Therefore the required solution will be

$y.\,e^{\sin x}=\int\left(c_1x-\frac{1}{4}sin2\ x+c_2\right)\,e^{\sin x}\,dx+c_3$

$y.\,e^{\sin x}=\int c_1x+c_2)e^{\sin x}\,dx-\frac{1}{2}\int\sin x\,cosx\,e^{\sin x}\,dx+c_3$

Again substitute $\sin x=t$ in second integral of right side

$\frac{1}{2}\int\sin x\,cosx\,e^{\sin x}\,dx=\frac{1}{2}\int te^tdt=\frac{1}{2}(te^t-e^t)$

$=\frac{1}{2}(sinx-1)e^{\sin x}$

$\therefore\ y.\,e^{\sin x}=\int(c_1+c_2\,x)e^{\sin x}\,dx-\frac{1}{2}(sinx-1)e^{\sin x}+c_3$

Q.6 **Solve** $x^3\frac{d^3y}{dx^3}+2x\frac{d^1y}{dx^1}+2y=10\left(x+\frac{1}{x}\right)$

Solution substitute $z=\log e^x \Rightarrow x= e^z$

Therefore the given equation can be written in form

$[D(D-1)(D-2)+2D(D-1)+2]y=10(e^z+ e^{-z})$

Or $(D^3-D^2+2)y=10(e^z+ e^{-z})$ where $D=d/dz$

Here the auxiliary Equation is

$m^3-m^2+2=0$ or (m+1) $(m^2-2m+2)=0$

$\therefore m=-1,1\pm i$

C.F. = $c_1\, e^{-z}+ e^z\,(c_2 Cosz+c_3\sin z)$

$=c_1\,x^{-1}+x\,[c_2 Cos\,(logx)+c_3\sin(logx)]$

P.I. = $\frac{1}{(D+1)\,(D^2-2D+2)}\quad 10\,(e^z+ e^{-z})$

$=10\left[\frac{1}{(D+1)\,(D^2-2D+2)}\, e^z+\frac{1}{(D+1)\,(D^2-2D+2)}\, e^{-z}\right]$

$=10\left[\frac{1}{2}\, e^z+\frac{1}{D+1}\,\frac{e^{-z}}{1+2+2}\right]$

$=10\left[\frac{1}{2}\, e^z+\frac{1}{5\,(D+1)}\, e^{-z}\right]=10\left[\frac{1}{2}e^z+\frac{e^{-z}}{5}\frac{1}{5\,(D-1+1)}.1\right]$

$10\left[\frac{1}{2}\, e^z+\frac{1}{5}\, ze^{-z}\right]=5x+2x^{-1}\log x$

Therefore the general solution will be

$y=c_1\,x^{-1}+x\,[c_2 Cos(logx)+c_3\sin(logx)]+5x+2x^{-1}\,logx$

Q. 7 **Solve** $tdx=(t-2x)dt$

$tdy=(tx+ty+2x-t)dt$

the given equation is $tdx=(t-2x)dt$ ------- (1)

$tdy=(tx+ty+2x-t)\,dt$ -------------(2)

Adding equation (1) and (ii)

$tdx + tdy = (tx + ty)dt$

Or $dx + dy = (x + y)dt$

Or $\frac{dx+dy}{x+y} = dt$ ----------------------------(3)

On Integration $\log(x + y) = t + log\ c_1$

Or $(x + y) = c_1 e^t$ ----------------(4)

Multiply equation (1) by t we can write

$t^2 \frac{dx}{dt} + 2tx = t^2$ or $\frac{d}{dt}(t^2 x) = t^2$ ------------(5)

On Integration Equation (5) $t^2 x = \frac{1}{3}t^3 + C_2$

Or $x = \frac{1}{3}t + C_2\ t^{-2}$ ----------------------(6)

From Equation (4) and (6)

$y = C_1\ e^t - \frac{1}{3}t - C_2\ t^{-2}$ ---------------(7)

Equation (6) and (7) both together give the solution of given equation.

Q. 8 **Solve** $(2x^2 + 3x)\frac{d^2y}{dx^2} + (6x + 3)\frac{dy}{dx} + 2y = (x + 1)e^x$

Solution Here $P_o = 2x^2 + 3x$ $P_1 = 6x + 3$ $P_2 = 2$ and $Q = (x + 1)\ e^x$

Now $P_2 - P_1' + P_0'' = 2 - 6 + 4 = 0$

Therefore the given equation is exact whose first integral will be

$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int (x + 1)\ e^x dx + C_1$

Or $(2x^2 + 3x)\ \frac{dy}{dx} + \{(6x + 3) - (4x + 3)\}y = xe^x + C_1$

Or $(2x^2 + 3x)\frac{dy}{dx} + 2xy = xe^x + C_1$ --------------(1)

Equation (1) is not exact equation but it is linear Equation of first order therefore we can write Equation (1)

$$\frac{dy}{dx} + \frac{2}{2x+3}\, y = \frac{e^x}{2x+3} + \frac{c_1}{x(2x+3)} \quad \text{---------------(2)}$$

Here I.F. = exp. $\left[\int \frac{2}{2x+3}\, dx\right] = exp.\, [\log(2x+3)]$

$= 2x + 3$

Therefore the required solution will be

$$y\,(2x+3) = \int \left(\frac{e^x}{2x+3} + \frac{c_1}{x(2x+3)}\right)(2x+3)dx + c_2$$

Or $y(2x+3) = \int \left(e^x + \frac{c_1}{x}\right) dx + c_2$

$\therefore\ y\,(2x+3) = e^x + c_1 log x + c_2$

Q. 9 **Solve** $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$

Solution Taking first two members we have

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)}$$

Or $\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dz}{z}$

Or $\frac{dx+dy}{\frac{1}{\sqrt{2}}\cos(x+y)+\frac{1}{\sqrt{2}}\sin(x+y)} = \frac{dz}{\frac{1}{\sqrt{2}}z}$

Or $\frac{dx+dy}{\sin(x+y)+\Pi/4} = \sqrt{2}\,\frac{dz}{z}$

$$\int cosec\,(x+y+\Pi/4)\,(dx+dy) = \sqrt{2}\int\frac{dz}{z}$$

$$\log\tan\frac{1}{2}\,(x+y+\Pi/4) = \sqrt{2}\,log z + \log c_1$$

$$tan\,\frac{1}{2}(x+y+\frac{\Pi}{4} = z^{\sqrt{2}}\,c_1 \quad \text{--------- (1)}$$

Again $\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$

$$\left[\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}\,dx+dy\right]=dx-dy$$

$$\log\left[\cos(x+y)+\sin(x+y)\right]=x-y+\log c_2$$

$\cos(x+y)+\sin(x+y)=c_2\,e^{x-y}$ ----------------------------(2)

Equation (1) and (2) both together give the solution of given Equation.

Q.10 $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=x\log x$

Solution Substitute $Z=\log e^x \Rightarrow x=e^z$ therefore the given equation can be written as

$[D(D-1)-D+2]y=ze^z$ where $D=\frac{d}{dz}$

Or $D^2-2D+2)y=ze^z$

Here the auxiliary equation is $m^2-2m+2=0$

Or $m=1\pm i$ C.F. $=e^z(c_1\cos z+c_2\sin z)$

$=x\,[c_1\cos(\log x)+c_2\sin(\log x)]$

Again P.I. $=\frac{1}{D^2-2D+2}.ze^z = e^z\frac{1}{(D+1)^2-2(D+1)+2}z$

$=e^z\frac{1}{D^2+1}Z=e^z(1-D^2+\cdots)z$

$=ze^z=x\log x$

Therefore the general solution of given equation

$y=x[c_1\cos(\log x)+c_2\sin(\log x)]+x\log x$

Q. 11 $\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{(x^2+y^2)}$

Solution Taking $x,-y,-z$ as multipliers

$$\frac{dx}{z(x+y)}=\frac{dy}{z(x-y)}=\frac{dz}{(x^2+y^2)}=\frac{xdx-ydy-zdz}{0}$$

$xdx-ydy-zdz=0$

Or $2xdx-2ydy-2zdz=0$

Or $d(x^2-y^2-z^2)=0$

On Integration $x^2 - y^2 - z^2 = c_1$

Again taking $y, x, -z$ as multipliers

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{(x^2+y^2)} = \frac{ydx+xdy-zdz}{0}$$

$\therefore\ ydx + xdy - zdz = 0$

Or $2ydx + 2xdy - 2zdz = 0$

Or $2d(xy) - d(z^2) = 0$

On integration $2xy - z^2 = c_2$--------------(2)

Equation (1) and (2) both together give the complete solution.

Q.12 $(2x^2 + 3x)\frac{d^2y}{dx^2} + (6x+3)\frac{dy}{dx} + 2y = (x+1)\,e^x$

Solution Here $P_0 = 2x^2 + 3x$ $P_1 = 6x + 3$ $P_2 = 2$

And $Q = (x+1)e^x$

Now $P_2 - P_1' + P_0'' = 2 - 6 + 4 = 0$

Therefore the given equation is exact whose first integral will be

$$P_0\frac{dy}{dx} + (P_1 - P_0')y = \int (x+1)e^x\,dx + c_1$$

Or $(2x^2 + 3x)\frac{dy}{dx} + \{(6x+3) - (4x+3)\}y = xe^x + c_1$

Or $(2x^2 + 3x)\frac{dy}{dx} + 2xy = xe^x + c_1$ ------------(1)

Which is not exact Equation but it is linear Equation of first order. Therefore its can be written as

$\frac{dy}{dx} + \frac{2}{2x+3}y = \frac{e^x}{2x+3} + \frac{c_1}{x\,(2x+3)}$ ----------------------(2)

Here I.F.= exp. $\left[\int \frac{2}{2x+3}dx\right] = exp.\,[\log(2x+3)] = 2x+3$

Therefore $y.\,(2x+3) = \int\left(\frac{e^x}{2x+3} + \frac{c_1}{x\,(2x+3)}\right)(2x+3)dx + c_2$

$\therefore\ y.\,(2x+3) = e^x + c_1 log x + c_2$

Q.13 **Solve** $x^3\frac{d^3y}{dx^3}-3x^2\frac{d^1y}{dx^1}+6x\frac{dy}{dx}=(loge^x)^2$

Solution substitute $z = loge^x \Rightarrow x = e^z$ in given equation

Therefore we can write the given equation as

$[D(D-1)(D-2)-3D(D-1)+6D]y = z^2$ where $D=\frac{d}{z}$

Or $[D^3-6D^2+11D]y = z^2$

Here the auxiliary Equation is $m^3-6m^2+11m=0$

$\therefore m=0,\ \ 3+i\sqrt{2},\ \ 3-i\sqrt{2},$

$\therefore$C.F. = $c_1+c_2\ e^{(3+i\sqrt{2})^z}+c_3\ e^{(3-i\sqrt{2})^z}$

$= c_1 + e^{3z}\left[A\cos(z\sqrt{2})+B\ sin\left(z\sqrt{2}\right)\right]$

$= c_1 + x^3\left[A\cos(\sqrt{2}logx)+B\ sin\left(\sqrt{2}\log x\right)\right]$

P.I. = $\frac{1}{D^3-6D^2+11D}\ z^2=\frac{1}{D}\ \frac{1}{\left\{11-6\left(D-\frac{D^2}{6}\right)\right\}}\ z^2$

$=\frac{1}{11}\left\{1-\frac{6}{11}\left(D-\frac{D^2}{6}\right)\right\}^{-1}\ \left(\frac{z^3}{3}\right)$

$=\frac{1}{11}\ 1+\frac{6}{11}\left(D-\frac{D^2}{6}\right)+\frac{36}{121}\left(D-\frac{D^2}{6}\right)^2$

$+\frac{216}{1331}\left(D-\frac{D^2}{6}\right)^3+\cdots\Bigg]\left(\frac{z^3}{3}\right)$

$=\frac{1}{33}\left[z^3+\frac{6}{11}(3z^2-z)+\frac{36}{121}(6z-2)+\frac{216}{1331}(6)\right]$

$=\frac{1}{33}\left[z^3+\frac{18}{11}z^2+\frac{150}{121}z+\frac{504}{1331}\right]$

P.I.= $\frac{1}{33}\left[(log\ x)^3+\frac{18}{11}(log\ x)^2+\frac{150}{121}(log\ x)+\frac{504}{1331}\right]$

Therefore the general solution will be

$y = c_1 + x^3\left[A\ Cos\left(\sqrt{2}\ logx\right)+Bsin\left(\sqrt{2}\ log\ x\right)\right]$

$+\frac{1}{33}\left[(log\ x)^3+\frac{18}{11}(log\ x)^2+\frac{150}{121}(log\ x)+\frac{504}{1331}\right]$

Q.14 **Solve** $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$

Solution Substitute $1+x=v$ in the given equation therefore the given equation can be written as $v^2\frac{d^2y}{dv^2}+v\frac{dy}{dv}+y=4\cos\log v$--------(1) which is Homogeneous linear differential Equation therefore assume $z=\log v \Rightarrow v=e^z$

Equation (1) can be written as

$[D(D-1)+D+1]y=4\cos\log e^z$ $\quad D=\frac{d}{dz}$

Or $(D^2+1)y=4\cos z$

Here the auxiliary equation is $m^2+1=0 \Rightarrow m=\pm i$

C.F. = $c_1\cos(z+c_2)$

$c_1\cos(\log v+c_2)$

$c_1\cos+\{c_2+\log(1+x)\}$

P.I. $=\frac{1}{D^2+1}4\cos z=4.\left(\frac{z}{2}\right)\sin z=2z\sin z$

P.I. $=2\log(1+x)\sin\log(1+x)$

Therefore the general solution of given equation will be

$y=c_1\cos\{c_2+\log(1+x)\}+2\log(1+x)\sin\log(1+x)$

Q.15 **Solve** $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z-a\sqrt{x^2+y^2+z^2}}$

Solution Taking first two member we have

$\frac{dx}{x}=\frac{dy}{y}$

On Integration $\log x=\log y+\log c_1 or\ x=c_1y$ ------(1)

Again x,y,z taking as multipliers

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}} = \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2 - az\sqrt{x^2 + y^2 + z^2}}$$

Substitute $x^2 + y^2 + z^2 = u^2$ in last two member

$$\frac{dy}{y} = \frac{dz}{z - au} = \frac{u\,du}{u^2 - auz}$$

Or $\frac{dy}{y} = \frac{dz}{z - au} = \frac{du}{u - az} = \frac{dz + du}{(1-a)(u+z)}$

Now taking first and last member

$$(1-a)\frac{dy}{y} = \frac{dz + du}{u + z}$$

On Integration $(1-a)\log y = log\,(u+z) - log c_2$

Or $u + z = c_2 y^{1-a}$

Or $\sqrt{x^2 + y^2 + z^2} + z = c_2 y^{1-a}$ -------------(2)

Equation (1) and (2) both together give the complete solution of given equation.

Q. 16 **Solve** $\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}$

Solution Taking last two member

$(y - z)dy = (y + z)dz$

Or $ydy - zdz - (ydz + zdy) = 0$

Or $d\left\{\frac{1}{2}(y^2 - z^2\right\} - d(yz) = 0$

On integration $\frac{1}{2}(y^2 - z^2\} - yz = \frac{c_1}{2}$

$\therefore\ y^2 - z^2 - 2yz = c_1$ ------------- (1)

Taking $1, y, z$ as multipliers

$$\frac{x\,dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$

Or $d\,(x^2 + y^2 + z^2) = 0$

On Integration $x^2 + y^2 + z^2 = c_2$---------(2)

Equation (1) and (2) both together give the complete solution of given equation.

Q.17 Solve $\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^1[Sin(y+2x)]}$

Solution taking first two member

$$\frac{dx}{1} = \frac{dy}{-2}$$

On Integration $x = \frac{-y}{2} + c1'$

Or $x + \frac{y}{2} = C1'$or $2x + y = c_1$ ----------(1)

Again $\frac{dx}{1} = \frac{dz}{3x^2 \sin(y+2x)}$

Or $3x^2 dx = \frac{dz}{\sin c_1}$

On Integration $x^3 = \frac{z}{\sin c_1} + c_2$

$x^3 = \frac{z}{\sin(2x+y)} + c_2$ ------------(2)

Equation (1) and (2) both together give the solution of given equation.

Q.18 Solve $\frac{dx}{x^1+y^1} = \frac{dy}{2xy} = \frac{dz}{z(x+y)}$

Solution $\frac{dx+dy}{x^2+y^2+2xy} = \frac{dz}{z(x+y)}$

or

$$\frac{dx+dy}{(x+y)^2} = dz$$

On integration $\log(x+y) = logz + logc_1$

$x + y = zc_1$ --------(1)

Now taking first two member

$\frac{dx}{x^2+y^2} = \frac{dy}{2xy}$ or $\frac{dy}{dx} = \frac{2xy}{x^2+y^2}$

Let $y = vx \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}.x + v \quad \frac{2v}{1+v^2} = \frac{x\,dv}{dx} + v$

Or $\frac{x\,dv}{dx} = \frac{2v}{1+v^2} - v \Rightarrow \frac{x\,dv}{dx} = \frac{v\,(1-v^2)}{1+v^2}$

$\Rightarrow \frac{1+v^2}{v(1-v^2)}\,dv = \frac{dx}{x} \Rightarrow \left(\frac{1}{v} + \frac{2v}{1-v^2}\right)dv = \frac{1}{x}\,dx$

Or $log v - log(1-v^2) = log x + log\, c_2$

$\Rightarrow log \frac{v}{1-v^2} = log x.c_2$

$\frac{v}{1-v^2} = xc_2 \Rightarrow \frac{y}{x^2-y^2} = c_2$

$\therefore y = (x^2 - y^2)c_2$-----------(2)

Equation (1) and (2) both together give the complete solution of given equation.

Q.19 **Solve** $\frac{dx}{y^2+yz+z^2} = \frac{dy}{z^2+zx+x^2} = \frac{dz}{x^2+xy+y^2}$

Solution $\frac{dx-dy}{y^2-x^2+z\,(y-x)} = \frac{dy-dz}{z^2-y^2+x\,(z-y)} = \frac{dz-dx}{x^2-z^2+y\,(x-z)}$

$\Rightarrow \frac{dx-dy}{(y-x)(y+x+z)} = \frac{dy-dz}{(z-y)(x+y+z)} = \frac{dz-dx}{(x-z)(x+y+z)}$

$\Rightarrow \frac{dx-dy}{(y-x)} = \frac{dy-dz}{(z-y)} = \frac{dz-dx}{(x-z)}$

Now taking first two member

$\frac{dx-dy}{(y-x)} = \frac{dy-dz}{-y+z}$

Or $\frac{dx-dy}{(x-y)} = \frac{dy-dz}{y-z}$

On Integration $\log(x-y) = \log(y-z) + log\,c_1$

$\frac{x-y}{y-z} = c_1$ --------------(1)

Now taking last two member

$\frac{dy-dz}{z-y} = \frac{dz-dx}{x-z}$

Or $\frac{dy-dz}{y-z} = \frac{dz-dx}{-x+z} \Rightarrow \frac{dx-dz}{x-z} = \frac{dy-dz}{y-z}$

$log\ (y-z)\ \ = log\ (x-z) + log\ c_2$

$\frac{y-z}{x-z} = c_2$ ---------(2)

Equation (1) and (2) together give the complete solution of given Equation.

Q. 20 **Solve** $sin^2 x\ \frac{d^2 y}{dx^2} = 2y$

Solution Divide the given Equation by $sin^2 x$

$\frac{d^2y}{dx^2} - 2y\ Cosec^2 x = 0$ --------(1) which is not exact equation multiply the above equation by $\cot x$

$\cot x \frac{d^2y}{dx^2} + 0\frac{dy}{dx} - 2\cot x\,.\,cosec^2 x.\,y = 0$ ---------------**(2)**

Here $= P_0 = cotx\ \ P_1 = \ 0\ \ P_2 = -2cotx.\,cosec^2 x.\,\theta = 0$

Now $P_2 - P_1' + P_0'' = -2\cot x\ cosec^2 x - 0 + 2\cot x\ \ cosec^2 x\ = 0$

Therefore equation (2) is exact Equation whose first integral will be

$P_0\frac{dy}{dx} + (P_1 - P_0')y = c_1$

Or $cot\ x\ \frac{dy}{dx} +\ (0 + cosec^2 x)y = c_1$

Or $\frac{dy}{dx} + \frac{cosec^2 x}{\cot x}.\,y = c_1 tanx$ ------------------(3)

Which is linear equation of first order here I.F. = exp. $\left\{\int \frac{cosec^2 x}{\cot x}\ dx\right\}$

$= exp.(-log\ cotx) = \tan x$

$\therefore\ y\tan x = c_1\int tan^2 xdx + c_2$

$= c_1\int(sec^2 x - 1)\ dx + c_2$

$\therefore\ y\ tanx\ = c_1(\tan x - x) + c_2$

Q.20 **Solve** $\frac{xdx}{y^2 z} = \frac{dy}{xz} = \frac{dz}{y^2}$

Solution Taking first two member

$\frac{xdx}{y^2z} = \frac{dy}{xz}$

Or $x^2 dx = y^2 dy$

On Integration $x^3 - y^3 = c_1$ -------------(1)

Now taking first and last member

$\frac{xdx}{y^2z} = \frac{dz}{y^2}$

$xdx = zdz$

$x^2 - z^2 = c_2$ -------------(2)

Equation (1) and (2) both together give the complete solution of given Equation.

Q.21 **Solve** $\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$

Solution $\frac{dx+dy}{(2+x+y)} = \frac{dz}{z}$

On integration $log\,(2 + x + y) = logz + logc_1$

Or $(2 + x + y) = zc_1$ ------------(1)

Again $\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z} = \frac{dx+dy+dz}{2+x+y+z}$

$\frac{dz}{z} = \frac{dx+dy+dz}{2+x+y+z}$

$\log z = log\,(x + y + z + 2) + logc_2$

$z = (x + y + z + 2)\,c_2$ ------------(2)

Equation (1) and (2) together give the complete solution of given equation.

Multiple Choice Questions

Q.1 *The Value of* $\frac{1}{f(Q)}\, x^m$ =?*when* $f(m) \neq 0$

(A) $\frac{x^m}{f(m)}$ (B) $\frac{x^m}{f(-m)}$ (C) $\frac{x^m}{f(m^2)}$ (D) None of these

Answer (A)

Q.2 For the differential Equation $\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$

Its one solution will be

(A) $2 + x + y = zc_1$ (B) $2 - x + y = c_1$ (C) $2 - x - y = zc_1$ (D) None of these

Answer (A)

Q.3 For the differential Equation $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ **here C.F. = ?**

(A) $c_1x^{-1} + c_2x^2$ (B) $c_1x^{-1} + c_2x^{-2}$ (C) $c_1x^{-1} - c_2x^{-2}$ **(D) None of these**

Answer (B)

Q.4 For the differential Equation $\frac{dy}{dx} + \frac{2}{2x+3}y = e^x$ ***here Integral Factor= ?***

(A) $2x - 3$ (B) $4x + 3$ (C) $2x + 9$ **(D)** $2x + 3$

Answer (D)

Q.5 $\frac{d^3y}{dx^3} + \cos x\frac{d^2y}{dx^2} - 2\sin x\frac{dy}{dx} - y\cos x = \sin 2x$

Here the value of Q= ?

(A) cosx (B) $-2\sin x$ (C) $\sin 2x$ **(D)** $-y\cos x$

Answer (c)

Q.6 $(2x^2+3x)\frac{d^2y}{dx^2} + (6x+3)\frac{dy}{dx} + 2y = (x+1)e^x$

Here the value of $P_2 - P'_1 + P''_0$ =?

(A) 2 (B) 0 (C) 4 **(D) None of these**

Answer (B)

Q.7 For the differential Equation $\frac{dy}{dx} + \frac{\operatorname{cosec}^2 x}{\cot x}.y = c_1\tan x$ ***here the Integral factor =?***

(A) tanx (B) cot x (C) log tanx **(D)** $-\cot x$

Answer (A)

Q.8 ***The Value of*** $\frac{1}{D^4}(4x^3) = ?$

(A) 24 *(B) 0* *(C) 24x* ***(D) None of these***

Answer (B)

Q.9 ***The Value of*** $(D^2 + 3D^2)x^2 = ?$

(A) 8 *(B) 12* *(C) 6* *(D) None of these*

Answer (A)

Q.10 ***For the differential Equation*** $\frac{dy}{dx} - \frac{1}{x^1} \cdot y = 1 + \frac{2}{x^3} + \frac{c_1}{x}$ ***here integral factor = ?***

(A) $e^{-1/x}$ *(B)* e^{x} *(C)* $e^{1/x}$ *(D) None of these*

Answer (C)

Unit-IV

Linear differential Equation of second order

An Equation of the form

$\frac{d^2y}{dx^2} + P(x)\ \frac{dy}{dx} + Q(x)\ y = R(x)$ is called a linear differential equation of the second order, where P,Q and R are functions of x alone (or perhaps constants).

To find one integral belonging to the C.F. by inspection :- The given Equation

$$\frac{d^2y}{dx^2} + P\ \frac{dy}{dx} + Qy = R \text{ --------(1)}$$

1) $y = e^x$ will be a C.F. of equation (1) if $1 + p + Q = 0$
2) $y = e^{-x}$ will be a C.F. of equation (1) if $1 - P + Q = 0$
3) $y = e^{mx}$ will be a C.F. of equation (1) if $m^2 + pm + Q = 0$
4) $y = x$ will be a C.F. of equation (1) if $P + Qx = 0$
5) $y = x^2$ will be a C.F. of equation (1) if $2 + 2Px + Qx^2 = 0$
6) $y = x^m$ will be a C.F. of equation (1) if $m(m-1) + Pmx + Q = 0$

Removal of the first derivative or change of dependent variable :-

Equation $\frac{d^2y}{dx^2} + P\ \frac{dy}{dx} + Qy = R$

When $P + Qx \neq 0$, $1 - P + Q \neq 0$, $1 + P + Q \neq 0$

$$u = C.F. = exp.\left\{\frac{-1}{2}\int Pdx\right\}$$

Complete solution $y = vu$

$$\frac{d^2v}{dx^2} + I.v = S$$

Where $I = Q - \frac{1}{4}\ P^2 - \frac{1}{2}\frac{dp}{dx}$

And S $= R\ exp.\left\{\frac{1}{2}\int Pdx\right\} = \frac{R}{u}$

Change of Independent variable :-

Let the linear Equation of second order be

$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ where P,Q and R are functions of x. Let the independent variable x be changed to z. where z is a suitable function of x.

Substitute $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ these values in equation (1) yields

$$\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1.y=R_1$$

Where $P_1=\frac{\frac{d^2z}{dx^2}+\frac{Pdz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ $Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}$

$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}$ Here P_1, Q_1, R_1 will be functions of x and may be converted as function of z

By the chosen relation between z and x

Solution by means of operational factors :- Let the linear equation of second order be given as

$$P_0\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=R$$

In symbolic manner above equation may be written as $(P_0D^2+P_1D+P_2)y=R$ where $D=\frac{d}{dx}$ or $f(D)y=R$

Here $f(D)$ can be resolved into a product of two factors $f_1(D)$ and $f_2(D)$ such that $f_1(D)$ operates upon y and $f_2(D)$ operates upon the result of this operation. The result obtained in this way will be same as obtained in the case when $f(D)$ operates upon y.

Symbolically we can write

$$f(D)y=f_2(D)\{f_1(D)y\}$$

$$f(D)y=f_2(D)f_1(D)y$$

Q.1 **Solve** $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0$

Solution The given equation is $\frac{d^2y}{dx^2}-\left(2-\frac{1}{x}\right)\frac{dy}{dx}+\left(1-\frac{1}{x}\right)y=0$ ----------(1)

Here $P=-\left(2-\frac{1}{x}\right)$ $Q=1-\frac{1}{x}$

Here $1+P+Q=0$ hence $y=e^x$ will be a part of complementary function.

Now taking $y=ve^x$

$\frac{dy}{dx}=ve^x+e^x\frac{dv}{dx}$, $\frac{d^2y}{dx^2}=ve^x+2e^x\frac{dv}{dx}+e^x\frac{d^2y}{dx^2}$

Substitute the value of $\frac{dy}{dx},\frac{d^2y}{dx^2}$ in equation (1), we obtain

$e^x\left[\frac{d^2y}{dx^2}+2\frac{dv}{dx}+v\right]-\left(2-\frac{1}{x}\right)e^x\left(v+\frac{dv}{dx}\right)+\left(1-\frac{1}{x}\right)ve^x=0$

Or $\frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=0$ --------------(2)

Again we consider $\frac{dv}{dx}=P\Rightarrow\frac{d^2v}{dx^2}=\frac{dP}{dx}$

From equation (2) $\frac{dP}{dx}+\frac{P}{x}=0$ or $\frac{dP}{P}=\frac{-dx}{x}$

On Integration, $logp=-logx+logC_1$

$\therefore P=\frac{C_1}{x}\Rightarrow\frac{dv}{dx}=\frac{C_1}{x}\Rightarrow dv=\frac{C_1}{x}dx$

Again by Integration $v=C_1logx+C_2$

Therefore complete solution of given Equation is

$y=ve^x=(c_1logx+c_2)e^x$

Q.2 **Solve** $(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x$

Solution The given Equation is $\frac{d^2y}{dx^2}-\left(\frac{2x+5}{x+2}\right)\frac{dy}{dx}+\left(\frac{2}{x+2}\right)y=\left(\frac{x+1e^x}{x+2}\right)$

Here P$=-\frac{2x+5}{x+2}$ and Q$=\frac{2}{x+2}$

Here $m^2+mp+Q=0$, for $m=2, i.e. 2^2+2P+Q=0$

Therefore $y = e^{2x}$ will be a part of C.F. now taking

$y = ve^{2x}, \frac{dy}{dx} = 2ve^{2x} + e^{2x}\frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = 4ve^{2x} + 4e^{2x}\frac{dv}{dx} + e^{2x}\frac{d^2v}{dx^2}$

Substitute the value of dy/dx and d^2y/dx^2 in (1) we obtain

$\frac{d^2v}{dx^2} + \left(\frac{2x+3}{x+2}\right)\frac{dv}{dx} = \frac{(x+1)}{(x+2)}^{e^{-x}}$ -------------(2)

Again we consider $\frac{dv}{dx} = P$ then $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

From (2) $\frac{dp}{dx} + \left(\frac{2x+3}{x+2}\right) P = \frac{(x+1)}{(x+2)}^{e^{-x}}$ -------(3) which is

Linear Equation of first order whose

I.F. $= exp.\left\{\int \frac{2x+3}{x+2}\, dx\right\} = exp.\left\{\int \left(2 - \frac{1}{x+2}\right) dx\right\}$

$= exp.\{2x - \log(x+2)\} = \frac{e^{2x}}{x+2}$

Therefore solution of equation (3) will be

$P.\frac{e^{2x}}{x+2} = \int \frac{(x+1)^{e^{-x}}}{x+2}\frac{e^{2x}}{x+2}\, dx + C_1 = \int \frac{x+1}{(x+2)^2} e^x dx + C_1$

$= \left\{\frac{1}{x+2} - \frac{1}{(x+2)^2}\right\} e^x \quad dx + C_1$

$\frac{e^x}{x+2} + \int \frac{e^x}{(x+2)^2}\, dx - \int \frac{e^x}{(x+2)^2} dx + C_1$

$= \frac{e^x}{x+2} + C_1$

$\therefore P = \frac{dv}{dx} = e^{-x} + C_1(x+2)e^{-2x}$

Again by integration we obtain

$v = \int e^{-x} dx + C_1\,(x+2)\, e^{-2x} dx + C_2$

Or $v = -e^{-x} + C_1 \left\{\frac{(x+2)\, e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} dx\right\} + C_2$

Or $v = -e^{-x} + C_1 \left\{\frac{-(x+2)\, e^{-2x}}{2} - \frac{e^{-2x}}{4}\right\} dx + C_2$

$$v = -e^{-x}\ \frac{-1}{4}\ C_1\ (2x+5)\ e^{-2x} + C_2$$

Therefore the complete solution of given equation is

$$y = ve^{2x} = -e^{x} - \frac{1}{4}\ C_1\ (2x+5) + C_2 e^{2x}$$

Q.3 Solve $Sin^2 x \frac{d^2y}{dx^2} = 2y$ given that $y = \cot x$ is its one solution.

Solution The given equation in standard form is written as $\frac{d^2y}{dx^2} - \frac{2}{sin^2 x}.y = 0$ ----------(1)

Its given that $y = cotx$ is part of C.F.

Now taking $y = vcot\ x$ then $\frac{dy}{dx} = cotx\frac{dv}{dx} - vcosec^2 x$

And $\frac{d^2y}{dx^2} = cotx\frac{d^2v}{dx^2} - 2\ cosec^2 x\frac{dv}{dx} + 2vcosec^2 x\cot x$

Substitute the value of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in Equation (1)

We obtain $\frac{d^2v}{dx^2} - \frac{2}{sinx\ cosx}\ \frac{dv}{dx} = 0$ ---------(2)

Again we consider $\frac{dv}{dx} = P$, then $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

From Equation (2) $\frac{dP}{dx} - \frac{2}{sinx\ cosx}\ P = 0$

Or $\frac{dP}{P} = 4\ cosec2\ xdx$

On integration we obtain $logP = 4.\frac{1}{2}\log\tan x + log\ C_1$

Or $P = C_1\ tan^2 x \Rightarrow \frac{dv}{dx} = C_1 tan^2 x$

Again by integration $v = c_1 \int(sec^2 x - 1)\,dx + C_2$

$$v = c_1(\tan x - x) + c_2$$

Therefore the complete solution will be

$y = vcotx = c_1 - c_1 x\cot x + c_2 cotx$

Q.4 **Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1) = -3e^{x^2}\sin 2x$**

Solution Here $P = -4x$ $Q = 4x^2 - 1$ $R = -3e^{x^2}\sin 2x$

To remove the first derivative, we will choose

$y_1 = exp.\left\{\frac{-1}{2}\int -4x dx\right\} = e^{x^2}$

Putting $y = ve^{x^2}$, the equation will take the form

$\frac{d^2v}{dx^2} + Iv = S$ ---(1) where $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$

Or $I = (4x^2 - 1) - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1$

And $S = Rexp\left\{\frac{1}{2}\int pdx\right\}$

$= -3e^{x^2}sin2x.\,exp.\left\{\frac{1}{2}\int -4xdx\right\}$

$= -3e^{x^2}sin2x.\,e^{-x^2} = -3sin2x$

Therefore from Equation (1) $\frac{d^2v}{dx^2} + v = -3sin2x$----(2)

Equation (2) can be written as $(D^2 + 1)v = -3\sin 2x$ where $D = \frac{d}{dx}$

Here Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

C.F. $= c_1cosx + c_2sinx$

Again P.I. $= \frac{1}{D^2+1}(-3sin2x) = \frac{-3\sin 2x}{-2^2+1} = sin2x$

$\therefore v = c_1cosx + c_2sinx + sin2x$

Therefore the general solution of given Equation

$y = vu = (c_1cosx + c_2sinx + sin2x)e^{x^2}$

Q.5 **Solve:** $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + \left(a^2 y = \frac{1}{x^2}\right)$

Solution Here the given equation is

$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6} y = \frac{1}{x^8}$ ------(1)

Here $P = \frac{3}{x}$ $\quad Q = \frac{a^2}{x^6}$ $\quad R = \frac{1}{x^8}$

Changing the independent variable from x to z, the above equation will take the form

$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ -----------(2)

Where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ $\qquad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$

And $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

Now z will be chosen in such a way that

$Q_1 = a^2[constant]$ This gives $Q_1 = \frac{a^2/x^6}{(dz/dx)^2} = a^2 \Rightarrow \frac{dz}{dx} = \frac{1}{x^3}$

And on integration $z = \frac{-1}{2}x^2$

Again $P_1 = \frac{\frac{-3}{x^4} + \frac{3}{x}\cdot\frac{1}{x^3}}{\frac{1}{x^6}} = 0$ $\; R_1 = \frac{1/x^8}{1/x^6} = \frac{1}{x^2}$

Substitute the value of P_1, Q_1 and R_1 in equation (2) we obtain

$\frac{d^2y}{dz^2} + a^2 y = \frac{1}{x^2}$

Or $(D^2 + a^2)y = -2z$ where $D = \frac{d}{z}$

Here the auxiliary Equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore C.F. = c_1 \cos az + c_2 \sin az$

Again P.I. $= \frac{1}{D^2 + a^2}(-2z) = \frac{1}{a^2}\left(1 + \frac{D^2}{a^2}\right)^{-1}(-2z)$

$= \frac{1}{a^2}\left(1 - \frac{D^2}{a^2} + \cdots\right)(-2z) = \frac{-2z}{a^2}$

$\therefore y = c_1 \cos az + c_2 \sin az - \frac{2z}{a^2}$

$y = c_1 cos\frac{a}{2x^2} - c_2 sin\frac{a}{2x^2} + \frac{1}{a^2x^2}$

Q.6 **Solve $3x^2\frac{d^1y}{dx^1} + (2 + 6x - 6x^2)\frac{dy}{dx} - 4y = 0$**

Solution The given Equation in symbolic form will be

$\{3x^2D^2 + (2 + 6x - 6x^2)D - 4\}y = 0$

Or $\{(3x^2D^2 + 6xD + 2D) - (6x^2D + 4)\}y = 0$

Or $\{D(3x^2D + 2) - 2(3x^2D + 2)\}y = 0$

$\therefore\ (D - 2)(3x^2D + 2)y = 0$ ------------- (1)

Now let $(3x^2D + 2)y = v$ ------------- (2)

Then by (1) $(D - 2)v = 0\ \ or\ \frac{dv}{dx} - 2v$

On integration $logv = 2x + logc_1$

Or $v = c_1e^{x2}$ $- - - (3)$

Now from Equation (2) and (3)

$(3x^2D + 2)y = c_1e^{2x}$

Or $3x^2\frac{dy}{dx} + 2y = c_1e^{2x}$

Or $\frac{dy}{dx} + \frac{2}{3x^2}.y = \frac{c_1e^{2x}}{3x^2}$ -- (4)

It is a linear differential equation of first order whose I.F. $= exp\left\{\frac{2}{3}\int x^{-2}dx\right\} =$

$exp.\left(\frac{-2}{3x}\right)$

Therefore the solution of equation (4) is $y = \frac{c_1}{3}e^{\frac{2}{3x}}\int x^{-2}e^{\{2x - 2/3x)\}}dx + c_2e^{(2/3x)}$

Q.7 **Solve $(x + 2)\frac{d^1y}{dx^1} - (2x + 5)\frac{dy}{dx} + 2y = (1 + x)e^x$**

Solution The given Equation in symbolic form will be

$\{(x + 2)\ D^2 - (2x + 5)\ D + 2\}\ y = (1 + x)\ e^x$

Or $\{(x + 2)\ D^2 - 2(x + 2)\ D - D + 2\}\ y = (1 + x)\ e^x$

Or $[(x + 2)\ D(D - 2) - (D - 2)]\ \ y = (1 + x)\ e^x$

$\therefore\ \{(x + 2)\ D - 1\}(D - 2)y = (1 + x)\ e^x$ -----①

Now let $(D - 2)y = v$ --------②

Then from (1) $\{(x+2)D-1\}v=(1+x)e^x$

Or $(x+2)\frac{dv}{dx}-v=(1+x)e^x$

Or $\frac{dv}{dx}-\frac{1}{x+2}.v=\left(\frac{1+x}{x+2}\right).e^x$ ------③

Whose I.F. = $e\int -\frac{1}{x+2}dx = e^{-\log(x+2)}=\frac{1}{x+2}$

Therefore the solution of Equation (3) will be

$v.\frac{1}{x+2}=\int\frac{1+x}{(x+2)^2}.e^x\,dx+c_1=\int\frac{(x+2-1)}{(x+2)^2}.e^x dx+c_1$

$=\int\left(\frac{1}{x+2}-\frac{1}{(x+2)^2}\right)e^x dx+c_1$

$=\frac{e^x}{x+2}+c_1$ $\therefore \int e^x[f(x)+f'(x)]dx=e^x f(x)$

$\therefore v= e^x+c_1(x+2)$---------④

Substitute the value of v in equation (2)

$(D-2)y=e^x+c_1(x+2)$ or $\frac{dy}{dx}-2y=e^x+c_1(x+2)$-----⑤

Here I.F. $=exp.\int -2dx=e^{-2x}$

Therefore the solution of Equation ⑤ will be

$ye^{-2x}=\int\{e^x+c_1(x+2)\}e^{-2x}dx+c_2$

$\int e^{-x}dx+c_1\int(x+2)e^{-2x}dx+c_2$

$=-e^{-x}+c_1\left(\frac{e^{-2x}(x+2)}{2}-\frac{e^{-2x}}{4}\right)+c_2$

$\therefore y=-e^x-\frac{1}{4}c_1(2x+5)+c_2e^{2x}$

Q.8 Solve $\frac{d^2y}{dx^2}+a^2y=secax$ by the method of variation of parameters.

Solution Complementary function that is solution of the equation $\frac{d^2y}{dx^2}+a^2y=0$ will be

$y=a'cosax+b'sinax$ ---(1)

Let us assume that the complete solution of the given equation be

$y=A\ cosax+B\ sinax$—(2)

Where A,B are function of x

$\therefore \frac{dy}{dx}=-Aa\ sinax+Ba\ cosax+cosax\ \frac{dA}{dx}+sinax\ \frac{dB}{dx}$

Suppose $A_1 cosax + B_1 sinax = 0$ ----③

Then $\frac{dy}{dx} = -Aa\ sinax + Ba\ cosax$

And $\frac{d^2y}{dx^2} = -Aa^2 cosax - Ba^2 sinax - \frac{dA}{dx} a\ sinax + \frac{dB}{dx} a\ cosax$

Now substitute the value of y from (1) and its derivative in given equation we obtain

$-A_1 sinax + B_1 acosax = secas$ --------④

From equation (3) and (4), we obtain

$A_1 = \frac{dA}{dx} = \frac{-1}{a} tanax,\ B_1 = \frac{dB}{dx} = \frac{1}{a}$

On Integration $A = -\frac{1}{a}\int tanax\ dx + c_1 = \frac{1}{a^2}\log cosax + c_1$

And $B = \frac{1}{a}\int dx + c_2 = \frac{x}{a} + c_2$

Therefore the complete solution of given equation will be

$$y = \left(\frac{1}{a^2}\log cosax + c_1\right) cosax + \left(\frac{x}{a} + c_2\right) sinax$$

$y = c_1 cosax + c_2 sinax + \frac{x}{a} sinax + \frac{1}{a^2} cosax. \log cosax$

Q.9 Solve by the method of variation of parameters :

$\boldsymbol{x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x}$

Solution To find out the complementary function we will first solve this equation

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ ------① which is Homogenous

Equation of second order therefore substitute

$z = logx \Rightarrow x = e^z$, we obtain

$\{D(D-1) + D - 1\}y = 0$ where $D = \frac{d}{dz}$

Or $(D^2 - 1)y = 0$

$\therefore y = ae^z + be^{-z} = ax + bx^{-1}$ ------②

$\therefore C.F. = ax + bx^{-1}$ where a and b are constant.

Suppose the complete solution of given equation is

$y = Ax + Bx^{-1}$, $\frac{dy}{dx} = A - Bx^{-2} + A_1x + B_1x^{-1}$ where A &B are function of x

Suppose $A_1x + B_1x^{-1} = 0$ ---------(3) then $\frac{dy}{dx} = \text{A} - \text{B}x^{-2}$ and

$\frac{d^2y}{dx^2} = A_1 - B_1x^{-2} + 2Bx^{-3}$

Now substitute the value of y and its derivative in given equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^x$

We obtain $A_1 - B_1x^{-2} = e^x$ -----------④

From equation (3) and (4)

$A_1 = \frac{dA}{dx} = \frac{1}{2}e^x$ and $B_1 = \frac{dB}{dx} = \frac{-1}{2}x^2e^x$

On Integration $A = \frac{1}{2}e^x + c_1$ and $B = \frac{-1}{2}\int x^2e^x dx + c_2 = \frac{-1}{2}e^x(x^2 - 2x + 2) + c_2$

Therefore the complete solution of given equation is

$$y = \left(\frac{1}{2}e^x + c_1\right)x + \left[\frac{-1}{2}e^x(x^2 - 2x + 2) + c_2\right]x^{-1}$$

$$= c_1x + \frac{c_2}{x} + e^x - \frac{e^x}{x}$$

Q.10 Solve by the method of variation of parameters

$$x^2\frac{d^2y}{dx^2} - 2x(1 + x)\frac{dy}{dx} + 2(1 + x)y = x^3$$

Solution The given equation in standard from is written as

$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x$ ---------①

For find out the complementary function we will first simplify this equation

$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = 0$ ---------②

Here $P = -\frac{2(1+x)}{x}$ and $Q = \frac{2(1+x)}{x^2}$

Here $P + Q.x = 0$ therefore $y = x$ is a part of C.F.

Suppose the solution of equation (2) is $y = vx$ --------③

Now substitute the value of y and its derivative in equation (2) we obtain

$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0$ ---------④ or $(D^2 - 2D)v = 0$ where $D = \frac{d}{dx}$

$\therefore v = a.e^{0x} + be^{2x} = a + be^{2x}$

Therefore the complement function of given equation will be

$y = (a + be^{2x})\ x = ax + bxe^{2x}$ ---------⑤

Where a and b are constant. Again let the complete solution of given equation is $y = Ax + B.xe^{2x}$ ---------------⑥ where A and B are function of x

$\therefore\ \frac{dy}{dx} = A_1 x + B_1 xe^{2x} + A + Be^{2x} + 2Bxe^{2x}$

Suppose $A_1 x + B_1 . xe^{2x} = 0$ ------------------⑦

Then $\frac{dy}{dx} = A + Be^{2x} + 2Bxe^{2x}$

And $\frac{d^2y}{dx^2} = A_1 + B_1 e^{2x} + 2Be^{2x} + 2xB_1\ e^{2x} + 4Bxe^{2x}$

Now substitute the value of y and its derivative in equation (1), we obtain

$A_1 + B_1(1 + 2x)\ e^{2x} = x$ -----------------(8)

Now solve equation (7) and (8), we obtain

$A_1 = \frac{dA}{dx} = \frac{-1}{2}$ and $B_1 = \frac{dB}{dx} = \frac{1}{2}\ e^{-2x}$

After integration we obtain $A = \frac{-1}{2} \int dx + c_1 = \frac{-1}{2} x + c_1$

And $B = \frac{1}{2}\ e^{-2x} dx + c_2 = \frac{-1}{4}\ e^{-2x} + c_2$

Therefore the complete solution of given equation is $y = \left(\frac{-1}{2} x + c_1\right)\ x + \left(\frac{-1}{4} e^{-2x} + c_2\right) xe^{2x}$

Q.11 Solve by the method of variation of parameters

$$(x + 2)\ \frac{d^2y}{dx^2} - (2x + 5)\frac{dy}{dx} + 2y = (1 + x)e^x$$

Solution The given equation in standard form is written as

$$\frac{d^2y}{dx^2} - \left(\frac{2x+5}{x+2}\right) \frac{dy}{dx} + \frac{2}{x+2} . y = \left(\frac{1+x}{x+2}\right)\ e^x \text{----------①}$$

For find out the complementary function first we solve given equation :

$\frac{d^2y}{dx^2} - \left(\frac{2x+5}{x+2}\right)\frac{dy}{dx} + \frac{2}{x+2}.y = 0$ -------------------②

Here $P = -\left(\frac{2x+5}{x+2}\right)$ and $Q = \frac{2}{x+2}$

Here $2^2 + 2P + Q = 0$ therefore $y = e^{2x}$ will be a part of C.F. suppose the solution of equation (2) is $y = ve^{2x}$ -----------③

Now substitute the value of y and its derivative in equation (2) we obtain

$\frac{d^2v}{dx^2} + \left(\frac{2x+3}{x+2}\right)\frac{dv}{dx} = 0$ ------------④

Again let $\frac{dv}{dx} = P$ then $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

Therefore from (4) $\frac{dp}{dx} + \frac{2x+3}{x+2}.P = 0$

$or\ \frac{dP}{P} = \left\{\frac{1}{x+2} - 2\right\}dx$

On integration $\log P = \log(x+2) - 2x + \log\alpha$

$\therefore P = \alpha(x+2)e^{-2x}$

$\Rightarrow \frac{dv}{dx} = \alpha(x+2)e^{-2x}$

Again on integration $v = -\alpha\frac{1}{4}(2x+5)e^{-2x} + \beta$

$\therefore y = \left\{\frac{\alpha}{4}(2x+5)\ e^{-2x} + \beta\right\} e^{2x}$

Or $y = a(2x+5) + b\ e^{2x}$ ------------⑤

Where a, b are constant $a = \frac{-\alpha}{4}\ b = \beta$

Again suppose the complete solution of given

equation is $y = A(2x+5) + Be^{2x}$-----------⑥

where A and B are function of x

$\therefore \frac{dy}{dx} = A_1(2x+5) + B_1e^{2x} + 2a + 2Be^{2x}$

Let $A_1(2x+5) + B_1e^{2x} = 0$ ----------------⑦

Then $\frac{dy}{dx} = 2A + 2Be^{2x}, \frac{d^2y}{dx^2} = 2A_1 + 2B_1e^{2x} + 4Be^{2x}$

Now substitute the value of y and its derivative in equation (1) we obtain

$2A_1 + 2B_1e^{2x} = \frac{1+x}{x+2} e^x$ ------------------⑧

Now solve equation (7) and (8) we obtain

$A_1 = \frac{dA}{dx} = \frac{(1+x)}{4(x+2)^2} e^x$

And $B_1 = \frac{dB}{dx} = \frac{(1+x)(2x+5)e^{-x}}{4(x+2)^2}$

On integration A = $-\int \frac{(1+x)}{4(x+2)^2} e^x \, dx$

$A = \frac{-1}{4}\left\{\frac{1}{x+2} - \frac{1}{(x+2)^2}\right\} e^x dx = \frac{-1}{4} . \frac{e^x}{x+2} + c_1$

And $B = \int \frac{(1+x)\ (2x+5)e^{-x}}{4(x+2)^2} dx = \frac{1}{4}\int \left\{2 - \frac{1}{x+2} - \frac{1}{(x+2)^2}\right\} e^{-x} dx$

$B = \frac{e^{-x}}{4}\left\{\frac{1}{x+2} - 2\right\} + c_2$

Therefore the complete solution of given equation is

$y = \left\{-\frac{e^x}{4(x+2)} + c_1\right\}(2x+5) + \left\{\frac{e^{-x}}{4}\left(\frac{1}{x+2} - 2\right) + c_2\right\} e^{2x}$

$or\ y = c_1(2x+5) + c_2e^{2x} - e^x$

Q.12 Solve $x^2\frac{d^1y}{dx^1} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$

Solution The given equation in standard form is written as

$\frac{d^2y}{dx^2} - \frac{2}{x}(1+x)\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x$ ---------------(1)

Here $P = \frac{-2}{x}(1+x)\ Q = \frac{2(1+x)}{x^2}$

Here $P + Qx = 0 \therefore C.F. = x$

Complete solution $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

And $\frac{d^2y}{dx^2} = \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx}$

Substitute the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equations (1) we obtain

$\frac{2dv}{dx} + \frac{xd^2v}{dx^2} - \frac{2}{x}(1+x)\left(v + x\frac{dv}{dx}\right)\ \frac{+2(1+x)}{x^2}vx = x$

Or $\frac{xd^2v}{dx^2} + \frac{dv}{dx}(2-2-2x) = x$

$\frac{d^2v}{dx^2} - \frac{2dv}{dx} = \perp$ ----------------②

Let $\frac{dv}{dx} = P\ ,\frac{d^2v}{dx^2} = \frac{dP}{dx}$

From (2) $\frac{dP}{dx} - 2P = \perp$

here I.F. $= e^{-\int 2dx} = e^{-2x}\ P.e^{-2x} = \int e^{-2x}.1dx$

or $Pe^{-2x} = \frac{-e^{-2x}}{2} + c_1$

$\frac{dv}{dx} = \frac{-1}{2} + c_1e^{2x}$

$dv = \left(\frac{-1}{2} + c_1e^{2x}\right)dx$

On integration $v = \frac{-1}{2}x + \frac{c_1}{2}e^{2x} + c_2$

Therefore the complete solution is $y = vx$

$y = \frac{-1}{2}x^2 + \frac{c_1}{2}xe^{2x} + c_2x$

Q.13 Solve $x\frac{dy}{dx}-y=(x-1)\left(\frac{d^2y}{dx^2}-x+1\right)$

Solution The given equation is

$$x\frac{dy}{dx}-y=(x-1)\frac{d^2y}{dx^2}-(x-1)^2$$

Or $(x-1)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=(x-1)^2$

Or $\frac{d^2y}{dx^2}-\frac{x}{x-1}\frac{dy}{dx}+\frac{1}{x-1}\,y=x-1$------------①

Here $P=\frac{-x}{x-1}$ $Q=\frac{1}{x-1}\therefore P+Qx=0$

$\therefore C.F.=x$ complete solution $y=vx$

$$\frac{dy}{dx}=\frac{dv}{dx}.x+v.1,\frac{d^2y}{dx^2}=\frac{dv}{dx}+\frac{d^2v}{dx^2}.x+\frac{dv}{dx}$$

Putting the value of $y,\frac{dy}{dx},\frac{d^2y}{dx^2}$ in equation (1)

$$x.\frac{d^2v}{dx^2}+\frac{2dv}{dx}-\left(\frac{x}{x-1}\right)\left(v+x\frac{dv}{dx}\right)+\frac{1}{x-1}vx=x-1$$

Or $\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(\frac{2}{x}-\frac{x}{x-1}\right)=x-1$

$$\frac{d^2v}{dx^2}+\frac{dv}{dx}\left(\frac{2}{x}-\frac{x}{x-1}\right)=x-1$$

Let $\frac{dv}{dx}=P,\frac{d^2v}{dx^2}=\frac{dP}{dx}$

$$I.F.=e^{\int\left(\frac{2}{x}-\frac{x-1+1}{x-1}\right)dx}=e^{2\log x-x+\log(x-1)}$$

I.F. $=\frac{x^2}{x-1}.e^{-x}$

$$P.\frac{x^2}{x-1}e^{-x}=\int\frac{x^2}{x-1}e^{-x}(x-1)dx+c_1$$

$$=-x^2e^{-x}+\int 2xe^{-x}\,dx+c_1$$

$$P.\frac{x^2}{x-1}e^{-x}=-x^2e^{-x}-2xe^{-x}-2(e^{-x})+c_1$$

$P = -(x-1) - 2\left(\frac{x-1}{x}\right) - 2\left(\frac{x-1}{x^2}\right) + \frac{c_1(x-1)}{x^2}e^x$

$dv = \left[-x + \perp - 2 + \frac{2}{x} - \frac{2}{x} + \frac{2}{x^2} + c_1 e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\right]dx$

On integration

$v = \frac{-x^2}{2} - x - \frac{2}{x} + c_1\frac{e^x}{x} + c_2 \therefore y = vx$

$y = \frac{-x^3}{2} - x^2 - 2 + c_1 e^x + c_2 x$

Q.14 $\frac{d^2y}{dx^2} - 2tanx\frac{dy}{dx} + 5y = 0$

Solution Here $P = -2tanx$ $Q = 5$ $R = 0$

Let $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\tan x\, dx}$

$u = \sec x$---------②

∴ complete solution is $y = vu$

Substitute the value of $y_1 \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation (1)

Its convert this equation $\frac{d^2v}{dx^2} + I.V = S$ --------------③

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ $S = Re^{\frac{1}{2}\int P dx}$

$I = 5 - \frac{1}{4}(4tan^2x) - \frac{1}{2}(-2sec^2x)$

Or $I = 5 - tan^2x + sec^2x \Rightarrow I = 6$ and $S = 0$

From equation (3) $\frac{d^2v}{dx^2} + 6.v = 0$

Or $D^2 + 6v = 0$ $A.E. = m^2 + 6 = 0 \Rightarrow m = \pm i\sqrt{6}$

$\therefore C.F. = c_1 cos\sqrt{6}x + c_2 sin\sqrt{6}x$

P.I. =0

$v = c_1 cos\sqrt{6}x + c_2 \sin \sqrt{6}x$

∴ complete solution will be

$y = vu = secx\ (c_1 cos\sqrt{6}x + c_2 \sin \sqrt{6}x$

Q.15 **Solve $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0$**

Solution Here $P = \frac{2}{x}\ \ Q = \frac{a^2}{x^4}\ \ \ R = 0$

Changing the independent variable from $x\ to\ z$ the above equation will take the form

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ ----------②

Where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$

$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

Now z will be chosen in such a way that

$a^2 = \frac{\frac{a^2}{x^4}}{\left(\frac{dz}{dx}\right)^2} \Rightarrow \left(\frac{dz}{dx}\right)^2 = \frac{1}{x^4}$

∴ $\frac{dz}{dx} = \frac{1}{x^2}$ and on integration $z = -x^{-1}$

Again $P_1\ \frac{-2x^{-3} + 2x^{-1}x^{-2}}{x^{-4}} = 0\quad R_1 = 0$

From equation (2) $\frac{d^2y}{dz^2} + a^2y = 0$

Or $(D^2 + a^2)y = 0$

A.E. $= m^2 + a^2 = 0 \Rightarrow m^2 = -a^2 \Rightarrow m = \pm ai$

C.F. = $c_1 cosaz + c_2 \sin az$

Therefore the complete solution will be

$y = c_1 cos\frac{a}{x} - c_2 \sin\frac{a}{x}$

Q.16 **Solve** $\frac{d^2y}{dx^2} + (1 - \cot x)\frac{dy}{dx} - y\, cotx$

Solution The given equation is $\frac{d^2y}{dx^2} + (1 - cotx)\frac{dy}{dx} - ycotx = sin^2 x$ ------------(1)

Here $P = 1 - \cot x$ and $Q = -\cot x$

Here $I - P + Q = 0$ therefore $y = e^{-x}$ will be a part of C.F.

Let $y = ve^{-x}$ is complete solution of ①

$\frac{dy}{dx} = e^{-x}\frac{dv}{dx} - ve^{-x}$

And $\frac{d^2y}{dx^2} = e^{-x}\frac{d^2v}{dx^2} - 2e^{-x}\frac{dv}{dx} + ve^{-x}$

Now substitute the value of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in equation ①

$\frac{d^2v}{dx^2} - (1 + \cot x)\frac{dv}{dx} = e^x sin^2 x$

Again suppose $\frac{dv}{dx} = P, \frac{d^2v}{dx^2} = \frac{dP}{dx}$

From (2) $\frac{dP}{dx} - (1 + \cot x)P = e^x sin^2 x$ ----------------③

Which is linear differential equation in P whose

I.F. $= exp.\{\int -(1 + cotx)dx\} = exp.(-x - logsin\, x) = \frac{e^{-x}}{\sin x}$

∴ solution of equation (3) will be

$P.\frac{e^{-x}}{\sin x} = \int e^x sin^2 x \frac{e^{-x}}{\sin x} dx + c_1$

$= \int sinxdx + c_1 = -cosx + c_1$

$\therefore\ P = \frac{dv}{dx} = -\sin x \cos x \, . \, e^{x} + c_1 sinx . e^{x}$

$= \frac{-1}{2} sin2x . e^{x} + c_1 sinx . e^{x}$

Again by integration

$v = \frac{-1}{2} \int e^{x}\ sin2x\ dx + c_1 \int e^{x} sinx\ dx + c_2$

$v = \frac{-1}{2} . \frac{1}{5}\ e^{x} (\sin 2\,x - 2cos\ 2x) + \frac{c_1}{2} e^{x}\ (sinx - cosx) + c_2$

Therefore the complete solution of given equation is $y = ve^{-x}$

$= -\frac{1}{10}\ (sin2x - 2cos2x) + \frac{1}{2} c_1 (sinx - cosx) + c_2 e^{-x}$

Q.17 **Solve** $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\ coslog(1+x)$

Solution The given equation in standard form is written as

$\frac{d^2y}{dx^2} + \frac{1}{1+x}\ \frac{dy}{dx} + \frac{y}{(1+x)^2} = \frac{4\,cos\,log(1+x)}{(1+x)^2}$ ------------①

Here $P = \frac{1}{1+x}$, $Q = \frac{1}{(1+x)^2}$ $R = \frac{4\ cos\,log(1+x)}{(1+x)^2}$

Changing the independent variable from x to z the above equation will take the form

$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ ----------------------------------②

Where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

Now z will be chosen in such a way that $Q_1 = 1\ (constant)$

$Q_1 = \frac{1/(1+x)^2}{\left(\frac{dz}{dx}\right)^2} \perp\ \Rightarrow \frac{dz}{dx} = \frac{1}{1+x}$ on integration $z = \log{(1+x)}$

Again $P_1 = \frac{-\frac{1}{(1+x)^2} + \frac{1}{1+x} . \frac{1}{1+x}}{\left(\frac{1}{1+x}\right)^2} = 0$

And $R_1 = \frac{4\cos\log(1+x)}{(1+x)^2}\left(\frac{1}{1+x}\right)^{-2} = 4\cos\log\,(1+x)$

Substitute the value of P_1, Q_1 and R_1 in Eq. (2)

$\frac{d^2y}{dz^2} + y = 4\,\cos\log\,(1+x)$ or $(D^2+1)y = 4\,\cos z$ where $D = \frac{d}{dz}$

A.E. = $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore C.F. = c_1 \cos\,(z + c_2)$

Again $P.I. = \frac{1}{D^2+1}\ 4\cos z = 4.\frac{z}{2}.\sin z$

$= 2z\sin z \therefore y = c_1\cos\,(z+c_2)\ + 2z\sin z$

$= c_1\cos\,\{\log\,(1+x) + c_2\} + 2\log(1+x) \times \sin\log\,(1+x)$

Q.18 Solve $\frac{d^2y}{dx^2} + (\tan x - 3\cos x)\frac{dy}{dx} + 2y\cos^2 x = \cos^4 x$

Solution: Here $P = \tan x - 3\cos x$, $Q = 2\cos^2 x$ $R = \cos^4 x$

Changing the independent variable from x to z, the above equation will take the form

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$-----------------------①

Where $R_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$ $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$ $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

Now z will be chosen in such a way that $Q_1 = 2$

$Q_1 = \frac{2\cos^2 x}{\left(\frac{dz}{dx}\right)^2} = 2 \Rightarrow \frac{dz}{dx} = \cos x$

Again $P_1 = \frac{-\sin x + (\tan x - 3\cos x)\cos x}{\cos^2 x} = -3$

And $R_1 = \frac{\cos^4 x}{\cos^2 x} = \cos^2 x$

Substitute the value of P_1, Q_1, R_1 in equation ①

$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = \cos^2 x$

Or $(D^2-3D+2)y = 1 - sin^2 x = 1 - z^2$ where $D = \frac{d}{dz}$

A.E. $= m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$

$\therefore C.F. = c_1 e^z + c_2 e^{2z}$

Again P.I. $= \frac{1}{(D-1)(D-2)} . (1 - z^2)$

$= \frac{1}{2}(1-D)^{-1}\left(1-\frac{D}{2}\right)^{-1}(1-z^2)$

$= \frac{1}{2}\left\{(1+D+D^2+...)\left(1+\frac{D}{2}+\frac{D^2}{4}+...\right)\right\}(1-z^2)$

$= \frac{1}{2}\left\{1+\frac{3}{2}D+\frac{7}{4}D^2+...\right\}(1-z^2)$

$= \frac{1}{2}\left\{(1-z^2)+\frac{3}{2}(-2z)+\frac{7}{4}(-2)...\right\}$

$= -\frac{1}{4}(2z^2+6z+5)$

$\therefore \quad y = c_1 e^z + c_2 e^{2z} - \frac{1}{4}(2z^2+6z+5)$

$y = c_1 e^{sinx} + c_2 e^{2sinx} - \frac{1}{4}(2sin^2 x + 6\sin x + 5)$

Q.19 $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \left(\frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}\right)y = 0$

Solution Here $P = \frac{1}{x^{1/3}}$ $Q = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2}$ $R = 0$

$u = exp.\left\{-\frac{1}{2}\int Pdx\right\} = exp.\left\{-\frac{1}{2}\int \frac{dx}{x^{1/3}}\right\}$

$= exp.\left(-\frac{3}{4}x^{2/3}\right)$

Substitute $y = vu$ in given equation then its convert

$\frac{d^2v}{dx^2} + Iv = S$ --------①

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dp}{dx}$

$I = \frac{1}{4x^{2/8}} - \frac{1}{6x^{4/8}} - \frac{6}{x^2} - \frac{1}{4}\frac{1}{x^{2/8}} - \frac{1}{2}\left(-\frac{1}{3x^{4/8}}\right) = \frac{6}{x^2}$ and S=Rexp. $\left\{\frac{1}{2}\int pdx\right\} = 0$

Therefore form (1) $\frac{d^2v}{dx^2} - \frac{6}{x^2}\ v = 0$

Or $x^2\frac{d^2v}{dx^2} - 6v = 0$ which is Homogeneous

Linear differential equation substitute $z = loge^x$

Or $x = e^z$

$\{D\ (D-1) - 2\}v = e^{2z}$

Or $(D^2 - D - 2)\ v = e^{2z}\ where\ D = \frac{d}{dz}$

A.E. $= m^2 - m - 2 = 0$ or $(m+1)\,(m-2) = 0$

$\Rightarrow m = -1{,}2 \Rightarrow C.F. = c_1x^{-1} + c_2x^2$

Again P.I. $=\frac{1}{(D+2)(D-2)}\ e^{2z} = \frac{1}{(2+1)\,(D-2)}e^{2z}$

$= \frac{1}{3}\ \frac{e^{2z}}{D-2} = \frac{e^{2z}}{3}\frac{1}{(2+1-2)} = \frac{1}{3}\ e^{2z}.Z$

$=\frac{x^2\,109\ x}{3} \therefore\ v = c_1x^{-1} + c_2x^2 + \frac{x^2\,109x}{3}$

Therefore the complete solution of given equation is

$y = vu = \left(c_1x^{-1} + c_2x^2 + \frac{x^2\,109x}{3}\right)109x$

Q.20 Solve by using the method of undetermined coefficient

$$\frac{d^1y}{dx^1} + 2x\frac{dy}{dx} + (x^2+5)y = x$$

Solution Here $P = 2x\ Q = x^2 + 5$ and $R = e^{-x^2/2}$

Taking $u = e^{-\frac{1}{2}\int pdx} = e^{-x^2/2}$

For the removal of the first derivative and substituting $y = uv,$ we get

$\frac{d^2 v}{dx^2} + I.v = S$ --------------①

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 4$ $\quad S = Re^{\frac{1}{2}\int p\,dx}$

Hence equation ① in its new form is

$\frac{d^2 v}{dx^2} + 4v = x$ --------- (2)

Its complementary function is C.F=$c_1 cos2x + c_2\ sin2x$

For the particular integral we take $v = A_0 + A_1 x$ and we obtain $A_0 = 0, A_1 = \frac{1}{4}$

Hence the complete solution is given by

$v = c_1 cos2x + c_2\ sin2x + \frac{x}{4}$

Therefore

$y = uv = e^{-x^2/2}\left[c_1 cos2x + c_2\ sin2x + \frac{x}{4}\right]$ is the required solution of given equation.

Multiple Choice Questions

Q.1 ***For the differential Equation*** $\frac{d^1 y}{dx^1} + p\frac{dy}{dx} + Qy$ ***=R here C.F.=*** e^x***if***

(A) $I + P + Q = O$

(B) $I - P + Q = O$

(C $P + Qx = O$

(D) $I + P - Q = O$

Q.2 ***Differential Equation*** $\frac{d^1 v}{dx^1} + I.v = S$ ***here the value of I =?***

(A) $Q + \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

(B) $Q + \frac{1}{4}P^2 + \frac{1}{2}\frac{dP}{dx}$

(C) $Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

(D) $Q + \frac{1}{4}P^2 + \frac{1}{4}\frac{dP}{dx}$

Answer (C)

Q.3 ***The value of*** $\frac{1}{D^1 + a^1}(-2z) = ?$

(A) $\frac{4z}{a^2}$ (B) $\frac{z}{a^2}$ (C) $\frac{2z}{a^2}$ (D) $\frac{-2z}{a^2}$

Answer (D)

Q.4 ***The value of*** $\frac{1}{(D+2)(D-2)}e^{2z}=?$

(A) $\frac{1}{3}e^{2z}.z$ (B) $\frac{1}{9}e^{2z}z$ (C) $\frac{1}{3}e^{2z}.z^2$ (D) $e^{2z}.z$

Answer (A)

Q.5 $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ ***here C.F. will be*** $y=x$ ***if***

(A) $I+P+Q=0$ (B) $I-P+Q=O$

(C) $P+Qx=O$ (D) None of these

Answer (C)

Q.6 ***The value of*** $\frac{1}{D^2+1}(-3\ sin2x)=?$

(A) $\sin 2x$ (B) $\cos 2x$ (C) $3\cos 2x$ (D) None of these

Answer (A)

Q.7 ***The value of*** $\frac{1}{(D^2+1)^2}\ sinx=?$

(A) $\frac{1}{8}x^2\sin x$ (B) $\frac{-1}{8}x^2 sinx$ (C) $\frac{1}{8}x^2\cos x$ (D) $\frac{1}{8}x^2 sin^2 x$

Answer (B)

Q.8 ***The value of*** $(D-2)\ sinx=?$

(A) $cos\,x+2sinx$ (B) $sinx-2cosx$

(C) $\cos x-2sin\ x$ (D) cosx-2 cos x

Answer (C)

Q.9 ***The value of R***$=?$ ***for the following differential Equation*** $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+\frac{a^2}{x^4}y=0$

(A) 0 (B) $\frac{2}{x}$ (C) $\frac{a^2}{x^4}$ (D) 1

Answer (A)

Q.10 For the differential Equation $\frac{d^2y}{dx^2} + \frac{1}{x^{1/3}}\frac{dy}{dx} + \frac{1}{4x^{1/3}}y = 0$ ***here I.F. =?***

(A) exp. $\left(\frac{3}{4}x^{2/3}\right)$

(B) exp. $\left(\frac{3}{4}x^{-2/3}\right)$

(C)exp $\left(\frac{9}{4}x^{2/3}\right)$

(D) exp $\left(\frac{-3}{4}x^{2/3}\right)$

Answer (D)

Unit-5

Partial differential equations of the first order, Homogeneous and non homogeneous linear partial differential equation.

Partial differential equation

Definition :- Partial differential equations are those which contain two or more independent variable. Partial differential coefficient of

Let x, y be two independent variables and z be the dependent variable. i.e. z be the function of x and y both. The derivatives of z with respect to x and y will be called the partial derivatives of z denoted by

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$$

Here the notations, we will adopt

$$P = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2}, \qquad S = \frac{\partial^2 z}{\partial x \partial y} \qquad t = \frac{\partial^2 z}{\partial y^2}$$

The order of the highest derivative of the dependent variable appearing in the equation is called the order of the equation.

Partial differential Equation of first order :- A differential equation in which only the first order derivatives appear will be called a partial differential equation of first order.

Eq. ① $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

② $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = m^2$

First order Partial differential equation in two independent variables :-

A first order P.d.e. in two independent variables x, y in its general form is given by $f(x, y, z, p, q) = 0$ (1) where f is a known function of its arguments.

(a) Non linear partial differential equation of first order – If the function f is not linear in P and q the equation (1) is said to non linear p.d.e.
e.g. $P^2 - q^2 - 1$----------②

(b) Quasi linear P.d.e. of first order (Lagrange's equation) :- If the function f is linear in P and q but not necessarily linear in the dependent variable z the equation (1) is called a $quasi$ linear p.d.e.

It written as $P\ (x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z)$

Or $Pp+Qq=R$ ---- ③ where P,Q,R are functions of $x, y\ and\ z$.

(b) Linear Partial differential Equation of first order :- A linear $p.d.e.$ of first order is of the form :

$$P\ (x,y)\frac{\partial z}{\partial x}+Q(x,y)\frac{\partial z}{\partial y}=R_1(x,y)z+R_2(x,y)$$

$$P\ (x,y)P+Q(x,y)q=R_1(x,y)z+R_2(x,y)$$

Where P, q and z all appear linearly with P, Q, R_1 and R_2 all function of x and y only.

Charpit's method :- the general method of solving the partial differential equations of first order linear or non-linear in two variables in commonly known as charpit's method.

$$\frac{dP}{\frac{\partial f}{\partial x}+P\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-P\frac{\partial f}{\partial P}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial P}}\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{df}{0}$$

This equation is called characteristic equation or charpit auxiliary equation of the differential equation $F\ (x,y,z,P,q)=0$

Linear partial differential equation with constant coefficient :- The general form of the L.P.D.E of n^{th} order is

$$\left(A_0\frac{\partial^n z}{\partial x^n}+A_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+\cdots+A_n\frac{\partial^n z}{\partial y^n}\right)$$

$$+\left(B_0\frac{\partial^{n-1} z}{\partial x^{n-1}}+B_1\frac{\partial^{n-1} z}{\partial x^{n-2}\partial y}+\cdots+B_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)+\ldots$$

$$\left(M_0\frac{\partial z}{\partial x}+M_1\frac{\partial z}{\partial y}+\right)\ldots+N_0 z=f(x,y)\ \text{------ ①}$$

where $A_0, A_1, A_2 \ldots\ldots A_n;\ B_0, B_1, \ldots\ldots B_{n-1}$;

M_0, M_1, N_0 are either constants or functions of x and y.

If the coefficients $A_0 \ldots\ldots N_0$ are constants then equation (1) is known as linear partial differential equation with constant coefficients or briefly its can be written as

$F(D, D')z = f(x, y)$-- ② where $D = \frac{\partial}{\partial x}, D' \frac{\partial}{\partial y}$

The solution of equation (2) consists of complementary function and particular integral. The complementary function is the solution of $F(D, D')z = 0$ containing n arbitrary functions. Particular integral will be the solution of (2) not involving any arbitrary function.

Homogeneous Equations :- A partial differential equation is called Homogeneous if all the derivatives appearing in the equation are of the same order or it is of the form.

Q.1 **Solve** $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

Solution The given equation is in the form of $Pp + Qq = R$ where $P = x(y^2 + z)$, $Q = -y(x^2 + z)$, $R = z(x^2 - y^2)$ lagrange's auxiliary equation of given equation will be

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \text{ -----(1)}$$

Taking $x, y, -1$ as multiplier we obtain

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{xdx + ydy - dz}{0}$$

$$\Rightarrow xdx + ydy - dz = 0$$

On integration $x^2 + y^2 - 2z = c_1$ ---------(2)

Again taking $1/x, 1/y, 1/z,$ as multipliers we obtain

$$\frac{dx/x + dy/y\ dz/z}{0} \Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integration $\log x + \log y + \log z = \log c_2$

$\therefore\ xyz = c_2$

Therefore general solution of given equation will be

$p(x^2 + y^2 - 2z, xyz) = 0$

Q.2 **Find the complete integral of the following partial differential equation.**

$$Pxy + Pq + qy = yz$$

Solution here $f(x, y, z, p, q) = Pxy + Pq + qy - yz = 0$ -- ①

$\therefore$ Charpit's auxiliary equation will be

$$\frac{dp}{py - py} = \frac{dq}{(px + q) - qy} = \frac{dz}{-P(xy + q) - q(P + y)} = \frac{dx}{-(xy + q)} = \frac{dy}{-(P + y)}$$

From first term $dP = 0$ or $P = c_1$ substitute the value of P in equation (1) we obtain

$$c_1xy + c_1q + qy - yz = 0 \Rightarrow q = \frac{yz - c_1xy}{c_1 + y}$$

And $dz = Pdx + qdy$

$$\Rightarrow dz = c_1dx + \frac{yz - c_1xy}{c_1 + y}dy$$

Or $(dz - c_1dx) = \frac{y\,(z - c_1x)}{c_1 + y}\, dy$

Or $\frac{dz - c_1dx}{z - c_1x} = \frac{y}{c_1 + y}dy = \left(1 - \frac{c_1}{c_1 + y}\right)dy$

On integration $\log(z - c_1x) = y - c_1\log(c_1 + y) + c_2$

Or $log\,(z - c_1x)\,(c_1 + y)^{c_1} = e^{y + c_2} = c_3e^y$

Which is complete integral of given equation.

Q.3 **Solve** $\frac{\partial^1 z}{\partial x^1} + \frac{\partial^1 z}{\partial x \partial y} - 6\frac{\partial^1 z}{\partial y^1} = ycosx$

Solution The given equation in symbolic form can be written as

$D^2 + DD' - 6D'^2)z = ycos\,x$

Here auxiliary equation will be $m^2 + m - 6 = 0 \Rightarrow m = -3, 2$

C.F. $= Z = \phi_1(y - 3x) + \phi_2(y + 2x)$

Particular integral will be given by

$$z = \frac{1}{D^2 + DD' - 6D'^2}\,(ycosx)$$

$= \frac{1}{(D+3D')(D-2D')}\ (ycosx)$

$= \frac{1}{(D+3D')} \int (c_1 - 2x)\ cosx\ dx$ where $c_1 = y + 2x]$

$= \frac{1}{D+3D'}\ [(c_1 - 2x)sinx - 2\ cosx]$

$= \frac{1}{D+3D'}\ [ysinx - 2\cos x]$ [substituting the value of c_1]

$= \int [(c_2 + 3x)\sin x - 2cosx\ ]\, dx$ where $(c_2 = y - 3x)$

$= -(c_2 + 3x)\ cosx + 3\sin x - 2\sin x$

$= -y\cos x + \sin x$ substituting the value of c_2

Therefore the completed integral will be

$z = \phi_1(\ y - 3x) + \phi_2\ (y + 2x) - y\cos x + \sin x$

Q.4 **Solve** $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = xy$

Solution Substituting $x = e^u$ and $y = e^v$ so that

$x\frac{\partial}{\partial x} \equiv D, \quad x^2 \frac{\partial^2}{\partial x^2} \equiv D(D-1)$

$y\frac{\partial}{\partial y} \equiv D', \quad y^2 \frac{\partial^2}{\partial y^2} \equiv D'(D'-1)$

Where $D = \frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v} = D'$ the equation is transformed

In $[D\,(D-1) - D'(D'-1)]z = e^{u+v}$

Or $(D - D')(D + D' - 1)\ z = e^{u+v}$

Which is a linear Equation with constant coefficients.

$z = \phi_1(\ u + v) + e^u\ \phi_2(\ u - v)$

$= \quad \phi_1(\log x + \log y) + x\phi_2(\ logx - logy)$

$= \quad f_1(xy) + x\ f_2\left(\frac{y}{x}\right)$

Particular integral $= \frac{1}{(D-D')(D+D'-1)} \; e^{u+v}$

$\frac{1}{(D-D')(1+1-1)} \; e^{u+v} \; \frac{1}{(D-D')} e^{u+v}$

$= u.\, e^{u+v} = xy \log x$

Thus the complete solution will be $z = f_1(xy) + x\, f_2\left(\frac{y}{x}\right) + xy \log x$

Q.5 Solve by Charpit's method :

$$\mathbf{2xz - Px^2 - 2qxy + Pq = 0}$$

Solution Here $f(x, y, z, P, q) = 2xz - Px^2 - 2qxy + Pq = 0$ -----①

Therefore charpit auxiliary equation will be

$$\frac{dP}{\frac{\partial f}{\partial x}+P\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-P\frac{\partial f}{\partial P}+q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{\partial f}{\partial P}} = \frac{dy}{\frac{\partial f}{\partial q}}$$

$$\frac{dP}{2z-2qy} = \frac{dq}{0} = \frac{dx}{x^2-q} = \frac{dy}{2xy-P} = \frac{dz}{Px^2+2xyq-2Pq} \text{------------②}$$

From the second term we have $dq = 0 \Rightarrow q = a$

Substitute the value of q in equation ①

$2xz - Px^2 - 2qxy + Pa = 0 \Rightarrow P = \frac{2x\,(z-qy)}{x^2-a}$

Substituting the value of P and q in $dz = Pdx + qdy$

We get $dz = \frac{2x\,(z-qy)}{x^2-a}\, dx + ady$

$\frac{dz-ady}{z-ay} = \frac{2x\,dx}{x^2-a}$

On integration $\log(z-ay) = log(x^2-a) + \log c \Rightarrow z - ay = c(x^2-a)$

Which is complete integral of given equation.

Q.6 **Solve $(y^2+z^2-x^2)p-2xyq-2xz=0$**

Solution The given equation is in form of $Pp+Qq=R$ where $P=y^2+z^2-x^2, Q=-2xy$ $R=-2xz$

Lagrange's auxiliary equation of given equation will be

$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz}$ -----------①

Taking last term of Equation ①

$\frac{dy}{y}=\frac{dz}{z}$ on integration logy $=logz+logc_1 \Rightarrow y/z=c_1$ -------②

Again x,y,z as multiplier

From (1) $\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2xz}=\frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)}$

From last two term

$\frac{dz}{z}=\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}$

On integration $\log z=\log(x^2+y^2+z^2)-log\, c_2$

$\therefore \frac{(x^2+y^2+z^2)}{z}\; c_2$

Therefore the general solution of given equation will be $\phi\left(\frac{y}{z},\frac{x^2+y^2+z^2}{z}\right)$

Q.7 **Find the surface passing through the two lines $z=x=0$ and $z-1=x-y=0$ and satisfying the differential equation $r-4s+4t=0$**

Solution The given Equation in symbolic form can be written as $(D^2-4DD'+4D'^2)z=0$ –(1)

Here the auxiliary equation will be $m^2-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

Therefore the general solution of given equation will be $z=\phi_1(y+2x)+x\phi_2(y+2x)$ -(2)

Since Equation (2) passes through the straight lines $z=x=0$ ---(3) and

$z-1=x-y=0$ ---- (4)

Therefore from equation (2) and (3) we obtain $\phi_1(y+2x)=0$ -------(5)

Therefore from equation (5) and (2) we get

$z = x\ \phi_2(y + 2x)$--------(6)

Now substitute $z = 1$ & $y = x$ in equation (6)

We get $1 = x\ \phi_2(3x) \Rightarrow \phi_2(3x) = 1/x$

Substitute $3x = \boldsymbol{t}$ we get $\phi_2(\boldsymbol{t}) = 3/\boldsymbol{t}$

$\Rightarrow\ \phi_2\ (2x + y) = \frac{3}{2x+y}$ ---------- (7)

Putting the value of ϕ_1 and ϕ_2 from equation (5) and (7) in equation (2), we obtain

$Z = x\left(\frac{3}{2x+y}\right)$ or $3x = z(2x + y)$

Q.8 **Solve** $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial y^2} = \frac{4x}{y^2} - \frac{y}{x^2}$

Solution Here the auxiliary equation is

$(D^2 - 4D'^2)z = \frac{4x}{y^2} - \frac{y}{x^2}$

A.E. $= m^2 - 4 = 0 \Rightarrow m = 2, -2$

C.F.$= \phi_1(y + 2x) + \phi_2(y - 2x)$

Again P.I. $= \frac{1}{D^2 - 4D'^2}\left(\frac{4x}{y^2} - \frac{y}{x^2}\right)$

$= \frac{1}{(D-2D')(D+2D')} = \left(\frac{4x}{y^2} - \frac{y}{x^2}\right)$

$= \frac{1}{D-2D'}\int\left\{\frac{4x}{(c_1+2x)^2} - \frac{c_1+2x}{x^2}\right\}dx$ [where $y = c_1 + 2x$

$= \frac{1}{D-2D'}\int\left\{\frac{2}{c_1+2x} - \frac{2c_1}{(c_1+2x)^2} - \frac{c_1}{x^2} - \frac{2}{x}\right\}dx$

$= \frac{1}{D-2D'}\left[log\ (c_1 + 2x) + \frac{c_1}{(c_1+2x)} + \frac{c_1}{x} - 2\log x\right]$

$\frac{1}{D-2D'}\left[\log y + \frac{y-2x}{y} + \frac{y-2x}{x} - 2\log x\right]$ [$\therefore c_1 = y - 2x$

$= \int\left[log\ (c_2 - 2x) + 1 - \frac{2x}{c_2-2x} + \frac{c_2-2x}{x} - 2 - 2\log x\right]dx$ [$\therefore$where $y = c_2 - 2x$

$$= \int \left[log\ (c_2 - 2x) - \frac{2x}{c_2-2x} + \frac{c_2}{x} - 3 - 2\log x\right]\ dx$$

$$= x.\ log\ (c_2 - 2x) + \int \frac{2x}{c_2-2x}dx - \int \frac{2x}{c_2-2x}dx + c_2 logx - 3x - 2\left\{xlog\ x - \right.$$
$xx2\ dx$

$$= x\ log\ (c_2 - 2x) + (c_2 - 2x)logx - 3x + 2x$$

$$= x \log y + y \log x - x$$

Therefore general solution of given equation will be

$$z = \phi_1(y + 2x) + \phi_2(y - 2x) + xlog\ y + y\log x - x$$

Q.9 **Solve $pcos\ (x + y) + q\ sin\ (x + y) = z$**

Solution Here $P = \cos(x + y)$ $Q = Sin\ (x + y)$ and R=z

So subsidiary equation will be $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin\ (x+y)} = \frac{dz}{z}$

First two terms gives

$$\frac{dx + dy}{\cos(x+y)+sin\ (x+y)} = \frac{dx - dy}{\cos(x+y) - sin\ (x+y)}$$

Or $\frac{-sin\ (x+y)+cos(x+y)(dx+dy)}{cos\ (x+y)+\sin(x+y)} = dx - dy$

Integrating

$$log\ \{cos\ (x + y) + sin\ (x + y)\} = x - y + log\ c_1$$

Or $[cos\ (x + y) + sin\ (x + y)]e^{y-x} = c_1$ ------①

Again $\frac{dx + dy}{\cos(x+y)+sin\ (x+y)} = \frac{dz}{z}$

Or $\frac{dx+dy}{\sqrt{2}\left\{\frac{1}{\sqrt{2}}\ cos\ (x+y)+\frac{1}{\sqrt{2}}\sin\ (x+y)\right\}} = \frac{dz}{z}$

Or $\frac{dx+dy}{\sqrt{2}\left[\left(\ x+y+\frac{\pi}{4}\right)\right]} = \frac{dz}{z}$

Integrating $\frac{1}{\sqrt{2}}\ \log tan\ \left[\frac{x+y}{2} + \frac{\pi}{8}\right] = \log z + logc_2$

Or tan $\left[\frac{x+y}{2}+\frac{\pi}{8}\right] z^{-\sqrt{2}} = c_2{}'$ ---(ii)

Hence the general solution of given equation will be

$$\left[\{cos\,(x+y)+sin\,(x+y)\}e^{y-x}, z^{-\sqrt{2}}tan\left\{\frac{x+y}{2}+\frac{\pi}{8}\right\}\right]=0$$

Q.10 Find the complete integral of the following equation by harpit's method.

$$\boldsymbol{P^2+q^2-2px-2qy+2xy=0}$$

Solution Here $f(x,y,z,p,q) = P^2+q^2-2px-2qy+2xy=0$ --- (1)

Therefore charpit auxiliary equation will be

$$\frac{dp}{-2p+2y}=\frac{dq}{-2q+2x}=\frac{dx}{2x-2p}=\frac{dy}{2y-2q}$$

Or $\frac{dp}{y-p}=\frac{dq}{x-p}=\frac{dx}{x-p}=\frac{dy}{y-q}-0$

$$\frac{dp+dq}{x+y-p-q}=\frac{dx+dy}{x+y-p-q}$$

$dp+dq=dx+dy$

On integration $p+q=x+y+a$

Or $(p-x)+(q-y)=a$ -------------(3)

Equation (1) can be written as

$(p-x)^2+(q-y)^2=(x-y)^2a$ ------------(4)

Substitute $p-x=P$ and $q-y=Q$ in Eq. (3) and (4)

We obtain $P+Q=a$ ----------------(5)

$P^2+Q^2=(x-y)^2$ ------------------(6)

$\therefore\ (P-Q)^2=P^2+Q^2-2PQ$

$= P^2+Q^2-\{(P+Q)^2-(P^2+Q^2)\}$

$=2(P^2+Q^2)-(P+Q)^2$

$=2\,(x-y)^2-a^2$

$\therefore P - Q = \sqrt{2(x-y)^2 - a^2}$ -----------(7)

Again by equation (5) and (7) we obtain

$P = \frac{1}{2}\left[a + \sqrt{2(x-y)^2 - a^2}\right] = p - x$

And $Q = \frac{1}{2}\left[a - \sqrt{\{2(x-y)^2 - a^2\}}\right] = q - y$

$p = x + \frac{1}{2}\left[a + \sqrt{\{2(x-y)^2 - a^2\}}\right]$

$q = y + \frac{1}{2}\left[a + \sqrt{\{2(x-y)^2 - a^2\}}\right]$

Substitute the value of p and q in $dz = pdx + qdy$ we obtain

$dz = \left(x + \frac{1}{2}a\right)dx + \left(y + \frac{1}{2}a\right)\ dy + \frac{1}{2}\sqrt{\{2(x-y)^2 - a^2\}}\ (dx - dy)$

Or $2dz = (2x + a\ )dx + (2y + a\ )dy + \sqrt{(t^2 - a^2)}\frac{dt}{\sqrt{2}}$

On integration we obtain [taking $\sqrt{2(x-y)} = t$

$c + 2z = x^2 + ax + y^2 + ay + \frac{1}{\sqrt{2}}\left[\frac{1}{2}t\sqrt{t^2 - a^2} - \frac{1}{2}a^2 \log\{t + (t^2 - a^2)\}\right]$

where $t = \sqrt{2(x-y)}$

Q.11 ***Solve*** $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Solution Here the auxiliary equation is

$m^2 - m - 2 = 0 \Rightarrow m = 2, -1$

$\therefore C.F = \phi_1(y + 2x) + \phi_2(y - x)$

Again P.I.$= \frac{1}{D^2 - DD' - 2D'^2}\{(y-1)e^x\}$

$= \frac{1}{(D-2D')(D+D')}\{(y-1)e^x\}$

$= \frac{1}{(D-2D')}\{(c_1 + x - 1)e^x dx\}$ [where $y = c_1 + x$

$= \frac{1}{D-2D'}\ [(c_1 + x - 2)e^x]$

$= \frac{1}{D-2D'}[(y-2)e^x]$ $[\therefore c_1 = y - x]$

$= \int (c_2 - 2x - 2)e^x dx$ where $y = c_2 - 2x$

P.I. $= (c_2 - 2x)e^x = ye^x$ $\therefore$ $c_2 = y + 2x$

Therefore $z = \phi_1(y + 2x) + \phi_2(y - x) + ye^x$

Q.12 **Solve $p + q = x + y + z$**

Solution The given equation is in form of $Pp + Qq = R$

Where $P = 1$ $Q = 1$ and $R = x + y + z$

Therefore Lagrange's auxiliary equation will be

$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x+y+z}$ -------①

From first two term we have $dx - dy = 0$

On integration $x - y = c_1$ -----②

Again from last two term we have

$\frac{dy}{1} = \frac{dz}{x+y+z}$ $\Rightarrow \frac{dy}{1} = \frac{dz}{L_1 + 2y + z}$

$\frac{dz}{dy} = c_1 + 2y + z \Rightarrow \frac{dz}{dy} - z = c_1 + 2y$ -------③

Which is linear equation in x and y whose $I.F. = e^{\int -1 dy} = e^{-y}$

$\therefore$ $z.e^{-y} = \int (c_1 + 2y)e^{-y} dy + c_2$

$= (c_1 + 2y)(-e^{-y}) - \int 2(-e^{-y})dy + c_2$

$= (-e^{-y})(c_1 + 2y) - 2e^{-y} + c_2$

$-e^{-y}(x - y + 2y + 2) + c_2$ [from (2)]

$\therefore$ $e^{-y}(x + y + z + 2) = c_2$ --------(4)

Therefore general solution of given equation is

$\phi\,[x - y, e^{-y}(x + y + z + 2)] = 0$

Q.13 **Solve $(D^2 - DD' - D' - 1)z = \sin(x + 2y)$**

Solution The given equation can be written as

$(D + 1)\ (D + D' - 1)z = \sin\ (x + 2y)$

$\therefore C.F. = e^{(-1)x}\phi_1(y + 0.x) + e^x\phi_2\{y + (-1)x\}$

$C.F. = e^{-x}\phi_1(y) + e^x\phi_1(y - x)$

Again P.I. = $\frac{1}{D^2 - DD' - D' - 1}\ \sin\ (x + 2y)$

P.I. = $\frac{1}{-(+1)^2 - 1.2 + D' - 1}\sin(x + 2y)$

P.I. =$\frac{1}{D' - 4}\ \frac{D' + 4}{D' + 4}\sin(x + 2y)$

P.I. =$\frac{D' + 4}{D'^2 + 16}\sin(x + 2y) = \frac{D' + 4}{-(2)^2 - 16}\sin\ (x + 2y)$

P.I.= $\frac{-1}{20}\ \{2\cos(x + 2y) + 4\sin(x + 2y)\}$

Therefore general solution of given equation will be

$z = e^{-x}\phi_1(y)$

Q. 14 **Solve $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = logx$**

Solution Suppose $x = e^u \Rightarrow u = logx$

And $y = e^v \Rightarrow v = \log y$

Using these substitution the given equation can be written as

$\{D(D - 1) - D'(D' - 1) + D - D'\}z = u$ where $D = \frac{\partial}{\partial u}$

Or $(D^2 - D'^2)z = u$

Or $\{(D - D')\ (D + D')\}z = u$ $D' = \frac{\partial}{\partial v}$

C.F. = $e^{0.u}\phi_1(v + u) + e^{0.u}\phi_2(v + u)$

$= \phi_1(logy + logx) + \phi_2(logy - logx)$

$= \phi_1 \{logxy)\} + \phi_2\{\log(y/x)\}$

C.F. $= f_1\ (xy) + f_2(yx)$

Again P.I. $= \frac{1}{D^1 - D'^1}\ u = \frac{1}{D^1}\left(1 - \frac{D'^1}{D^1}\right)^{-1} u$

$= \frac{1}{D^1}\left(1 + \frac{D'^1}{D^1} \pm \cdots\right)u = \frac{1}{D^1}u$

P.I. $= \frac{u^3}{6} = \frac{(logx)^3}{6}$

Therefore general solution of given equation will be $z = f_1\ (xy) + f_2\left(\frac{y}{x}\right) + \frac{(logx)^3}{6}$

Q.15 Solve the following equation by charpits method $(p^2 + q^2)\,Y = qz$

Solution Here $f(x, y, z, p, q) = (p^2 + q^2)\,Y - qz = 0$ (1)

Therefore charpit auxiliary equation will be

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-\frac{P\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

Or $\frac{dP}{-Pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}$ -----②

Taking first two term we have

$Pdp + qdq = 0$

On integration $p^2 + q^2 = a^2$-----------③

From 1 and 3, Eliminate the value of P, we have

$a^2y - qz = 0 \Rightarrow q = (a^2y/z)$

Putting the value of q in equation (3) we have

$P^2 + \frac{a^4y^2}{z^2} = a^2 \Rightarrow P = \frac{a}{z}\ z^2 - \sqrt{z^2 - a^2y^2}$

Substitute the value of P and q in $dz = pdx + qdy$

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\ dx + \frac{a^2y}{z}\ dy$$

$\Rightarrow \frac{zdz - a^2ydy}{\sqrt{z^2 - a^2y^2}} = adx$ --(4)

On integration

$(z^2 - a^2y^2)^{1/2} = ax + C$

$\Rightarrow\ z^2 - a^2y^2 = (ax + c)^2$ ------------(5) which is required complete integral

Put $C = \phi(a)$ in equation (5) we have

$z^2 - a^2y^2 = \{ax + \phi(a)\}^2$ which is generalized integral of given equation.

Q.16 **Find complete integral of given equation**

$\boldsymbol{z^2(p + q) = x^2 + y^2}$

Solution The given equation can be written as

$z^2p - x^2 = y^2 - z^2q = a$ (suppose)

$\Rightarrow P = \frac{a + x^2}{z^2}, q = \frac{y^2 - a}{z^2}$

Now dz $= Pdx + qdy$

$\frac{a + x^2}{z^2}dx + \frac{y^2 - a}{z^2}dy$

$\Rightarrow z^2dz = (a + x^2)dx + (y^2 - a)dy$

On integration $\frac{1}{3}z^3 = ax + \frac{1}{3}x^3 + \frac{1}{3}y^3 - ay + b$

$\Rightarrow z^3 = 3ax - 3ay + x^3 + y^3 + c$ where $c = 3b$

Which is required complete integral of given equation.

Q.17 **Solve** $\boldsymbol{\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x^2 \sin(x + y)}$

Solution The given equation can be written as

$(D^2 - DD' - 6D'^2)z = x^2 \sin(x + y)$

Or $(D-2D')(D+3D')z = x^2\sin(x+y)$

$\therefore$ C.F. $= e^{0.x}\phi_1(y+2x)\ e^{0.x}\phi_2(y-3x)$

C.F.$=\ \phi_1(y+2x)+\phi_2(y-3x)$

P.I =I.P. $\frac{1}{D^2-DD'-6D'^2}\left[x^2e^{i(x+y)}\right]$

$=\quad$ I.P $\quad e^{iy}\frac{1}{D^2+Di-6i^2}\ x^2e^{ix}$

$=\quad$ I.P $\quad e^{iy}.\,e^{ix}\frac{1}{(D+i)^2+(D+i)\ i+6}\ x^2$

$=\quad$ I.P $\quad e^{i(x+y)}\frac{1}{D^2+3iD+4}\ x^2$

$=\quad$ I.P. $\frac{e^{i(x+y)}}{4}\left\{1+\frac{D^2+3iD}{4}\right\}^{-1}(x^2)$

$=\quad$ I.P. $\frac{e^{i(x+y)}}{4}\left\{1-\frac{D^2+3iD}{4}+\left(\frac{D^2+3iD}{4}\right)^2-\cdots\ldots\right\}(x^2)$

$=$ I.P. $\frac{e^{i(x+y)}}{4}\left\{1-\frac{3iD}{4}-\frac{D^2}{4}-\frac{9}{16}D^2+\cdots(x^2)\right\}$

$=$ I.P. $\frac{e^{i(x+y)}}{4}\left\{x^2-\frac{3i}{4}-2x-\frac{13}{16}\ 2\right\}$

$=$ I.P. $\frac{1}{4}\left\{\cos(x+y)+i\sin(x+y)\times\left(x^2-\frac{13}{8}\right)-\frac{3x}{2}i\right\}$

$=\frac{1}{4}\left[\left(x^2-\frac{13}{8}\right)\sin(x+y)-\frac{3x}{2}\cos(x+y)\right]$

Therefore general solution of given equation will be

$$z=\phi_1(y+2x)+\phi_1(y-3x)+\frac{1}{4}\left[\left(x^2-\frac{13}{8}\right)\sin(x+y)-\frac{3x}{2}\cos(x+y)\right]$$

Q.18 **Solve the following equation by charpit's method : $p^2x+q^2y-2=0$**

Solution Here $f(x,y,z,p,q)=p^2x+q^2y-2=0$ -----(1)

Therefore charpit auxiliary equation will be

$$\frac{dp}{-P+P^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(P^2x+q^2y} = \frac{dx}{-2px} = \frac{dy}{-2qy}$$

$$\Rightarrow \frac{P^2\,dx+2px\,dp}{P^2x} = \frac{q^2dy+2qy\,dq}{q^2y}$$

$$\Rightarrow \frac{d(P^2x)}{P^2x} = \frac{d(q^2y)}{q^2y}$$

On integration $log(P^2x) = log(q^2x) + \log a$

$\Rightarrow P^2x = a\,q^2y$------(2)

From equation (1) and (2) we have

$a\,q^2y + q^2y - z = 0$

$$\Rightarrow q = \left\{\frac{z}{(a+1)y}\right\}^{1/2} \;\&\; P = \left\{\frac{az}{(1+a)x}\right\}^{1/2}$$

Substitute the value of p and q in $dz = pdx + qdy$ we obtain

$$dz = \left\{\frac{az}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(a+1)y}\right\}^{1/2} dy$$

$$\Rightarrow \sqrt{1+a}\,\frac{dz}{\sqrt{z}} \quad \sqrt{a}\,\frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}$$

On Integration

$\sqrt{\{(1+a)z\}} = \sqrt{ax} + \sqrt{y} + b$

Which is complete integral of given equation.

Q. 19 **Solve $z - xp - yq = a\sqrt{x^2 + y^2 + z^2}$______①**

Solution The given equation can be written as

$xp + yq = z - a\sqrt{x^2 + y^2 + z^2}$ which is in

The form of $Pp + Qq = R$ where

$P = x\; Q = y\; R = z - a\sqrt{x^2 + y^2 + z^2}$

Here lagrange's auxiliary equation will be

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}} = \frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2 - az\sqrt{x^2 + y^2 + z^2}} \text{ --------(2)}$$

Taking first two term we have $\frac{dx}{x} = \frac{dy}{y}$ on integration $log x = log y + log c_1$

$\therefore x/y = c_1$------------------(3)

Now taking third and forth term we have

$$\frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}} = \frac{x\,dx + y\,dy + z\,dz}{(x^2 + y^2 + z^2) - az\sqrt{x^2 + y^2 + z^2}}$$

Substitute $x^2 + y^2 + z^2 = u^2$

We have $\frac{dz}{z - au} = \frac{u\,du}{u^2 - azu}$

$$\Rightarrow \frac{dz}{z - au} = \frac{du}{u - az} = \frac{dz + du}{(1-a)(z+u)} \text{ --------(4)}$$

Now from Equation (2) and (4) we have

$$\frac{dx}{x} = \frac{dz + du}{(-a)(z+u)}$$

Or $(1 - a)\frac{dx}{x} = \frac{dz + du}{z + u}$

Integration $(1 - a)\ log x = \log(z + 4) - log c_2$

Or $z + u = c_2 x^{(1-a)}$

Or $\frac{z - \sqrt{x^2 + y^2 + z^2}}{x^{(1-a)}} = c_2$ ----------------(5)

Therefore general solution of given equation will be $\phi\left[\frac{x}{y}, \frac{z + \sqrt{x^2 + y^2 + z^2}}{x^{(1-a)}}\right] = 0$

Q.20 **Solve** $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$

Solution the given equation can be written as

$(D^2 + 3DD' + 2D'^2)z = x + y$

Here auxiliary equation is $m^2 + 3m + 2 = 0$

Or $(m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$\therefore C.F. = \phi_1(y-x) + \phi_2(y-2x)$

Again P.I.=$\frac{1}{D^2+3DD'+2D'^2}(x+y)$

P.I.=$\frac{1}{D^2\left(1+\frac{3D'}{D}+2\frac{D'^2}{D^2}\right)}(x+y)$

P.I.=$\frac{1}{D^2}\left(1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^{-1}(x+y)$

P.I.=$\frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\cdots\right](x+y)$

P.I.=$\frac{1}{D^2}(x+y) - \frac{3D'}{D^3}(x+y)$

P.I.=$\frac{1}{D^2}(x+y) - \frac{3}{D^3}(1) = \frac{1}{6}x^3 + \frac{1}{2}x^2y - 3\frac{x^3}{6}$

P.I. =$\frac{1}{2}x^2y - \frac{1}{3}x^3$

Therefore general solution of given equation will be

$z = \phi_1(y-x) + \phi_2(y-2x) + \frac{1}{2}x^2y - \frac{1}{3}x^3$

Q.21 **Solve $D^3 - 2D^2D' - DD'^2 + 2D'^3Z = e^{x-y}$**

Solution Here auxiliary equation is $m^3 - 2m^2 - m + 2 = 0$

Or $m-1\ m+1\ m-2 = 0 \Rightarrow m = 1, -1, 2$

$\therefore C.F. = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+2x)$

Again P.I. =$\frac{1}{D^3-2D^2D'-DD'^2+2D'^3}e^{x-y}$

P.I= $\frac{1}{(D-D')(D+D')}\left\{\frac{1}{(1-2)}\int e^4 d4\right\}$ where $u = x+y$

P.I= $\frac{-1}{(D-D')(D+D')}e^{x-y}$

P.I= $\frac{-1}{(D-D')}\left\{\frac{1}{(1+1)}\int e^4 d4\right\}$

P.I= $\frac{-1}{2}\frac{1}{D-D'}e^{x-y} = -\frac{1}{2}\frac{x}{1/1}e^{x-y}$

P.I. = $\frac{-1}{2}xe^{x-y}$

There general solution of given equation will be

$$z = \phi_1\ (y+x) + \phi_2\ (y-x) + \phi_3\ (y+2x) - \frac{1}{2}xe^{x+y}$$

Q.22 **Solve : $D(D-2D')(D+D')z = e^{x+2y}(x^2+4y^2)$**

Solution C.F. = $e^{0.x}\phi_1(y) + e^{0.x}\phi_2(y+2x) + e^{0.x}\phi_3(y-x)$

$= \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x)$

Again P.I. =$\frac{1}{D(D-2D')(D+D')}\{e^{x-2y}(x^2+4y^2)\}$

$= \frac{1}{D(D-2D')(D-D')}x^2e^{x+2y} + 4\frac{1}{D(D-2D')(D-D')}y^2e^{x+2y}$

$= P_1 + P_2$(lets suppose)

Now $P_1 = e^{2y}\frac{1}{D(D-4)(D-2)}\ x^2e^x$

$P_1 = e^{x+2y}\frac{1}{(D-1)(D-1-4)(D-1-2)}x^2$

$P_1 = e^{x+2y}\frac{1}{(D-1)(D^2-9)}x^2 = \frac{-1}{9}e^{x+2y}\ (1+D)^{-1}\left(1-\frac{D^2}{9}\right)^{-1}(x^2)$

$P_1 = \frac{-1}{9}e^{x+2y}\ (1-D+D^2 \ldots\ldots)\left(1+\frac{D^2}{9}+\cdots..\right)(x^2)$

$P_1 = \frac{-1}{9}e^{x+2y}\left(1-D+\frac{10}{9}D^2+\cdots\right)(x^2)$

$P_1 = \frac{-1}{9}e^{x+2y}\left(x^2-2x+\frac{20}{9}\right)$

And $P_2 = 4e^x\frac{1}{1(1-2D')(1-D')}y^2e^{2y}$

$P_2 = 4e^{x+2y}\frac{1}{\{1-2'(D'-2)\}(1-D'-2)}y^2$

$P_2 = -4e^{x+2y}\frac{1}{(3-2D')(3-D')}y^2$

$$P_2 = \frac{-4e^{x+2y}}{9}\left(1+\frac{2D'}{3}\right)^{-1}\left(1+\frac{D'}{3}\right)^{-1}(y^2)$$

$$P_2 = \frac{-4}{9}e^{x+2y}\left(1-\frac{2D'}{3}+\frac{4D'^2}{9}\ldots\right)\left(1-\frac{D'}{3}+\frac{D'^2}{9}\ldots\right)(y^2)$$

$$P_2 = \frac{-4}{9}e^{x+2y}\left(1-D'+\frac{7}{9}D'^2\right)(y^2)$$

$$P_2 = \frac{-4}{9}e^{x+2y}\left(y^2-2y+\frac{14}{9}\right)$$

Therefore general solution of given equation will be

$$z = \phi_1(y)+\phi_2(y+2x)+\phi_3(y-x)\frac{-1}{9}e^{x-2y}\left(x^2-2x+\frac{20}{9}\right)$$
$$\frac{-4}{9}e^{x+2y}\left(y^2-2y+\frac{14}{9}\right)$$

Or $z == \phi_1(y)+\phi_2(y+2x)+\phi_3(y-x)\frac{-e^{x+2y}}{81}\left(9x^2+36+\frac{20}{9}\right)$

Q.23 **Solve $(x-y)p+(x+y)q=2xz$**

Solution The given equation is the form of $Pp+Qq=R$

Where $P=(x-y)\ Q=(x+y)$ and $R=2xz$

Therefore lagrange auxiliary equation will be

$\frac{dx}{x-y}=\frac{dy}{x+y}=\frac{dz}{2xz}$ ---------(1)

Now taking first two term

$\frac{dy}{dx}=\frac{x+y}{x-y}$ which is Homogeneous

Equation of first order therefore substitute $y=vx$

$v+x\frac{dv}{dx}=\frac{x-vx}{x-vx}$ $\left[\because \frac{dy}{dx}=v+x\frac{dv}{dx}\right]$

Or $\frac{xdv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v^2}{1-v}$

Or $\frac{dx}{x}=\frac{1-v}{1+v^2}dv\ or\ \frac{dx}{x}=\frac{dv}{1+v^2}=\frac{v}{1+v^2}dv$

On integration $logx=tan^{-1}v\frac{-1}{2}log(1+v^2)+log\ c_2$

Or $2\log x = 2\tan^{-1}(y/x) - \log(1 + y^2/x^2) + \log c_2$

Or $\log x^2 + \log\left(\frac{x^2+y^2}{x^2}\right) - \log c_2 = 2\tan^{-1}\left(\frac{y}{x}\right)$

Or $\log\left(\frac{x^2+y^2}{c_2}\right) = 2\tan^{-1}\left(\frac{y}{x}\right)$

Or $x^2 + y^2 = c_2 e^{2\tan^{-1}\left(\frac{y}{x}\right)}$

$\therefore\ (x^2 + y^2)e^{-2\tan^{-1}(y/x)} = c_2$ ------------------------(2)

Now taking $\perp, \perp, -\perp/z$ as multipliers

From equation (1) $\frac{dx+dy-\frac{1}{1}dz}{0} \Rightarrow dxdy - \frac{dz}{z} = 0$ on integration

$x + y - \log z = c_2$ ---------(3)

Therefore general solution of given equation will be

$\phi\{(x^2 + y^2)e^{-2\tan^{-1}(y/x)}, x + y - \log z\} = 0$

Q.24 **Solve :$(mz - ny)p + (nx - lz)q = ly - mx$**

Solution The given equation is form of

$Pp + Qq = R$ where $P = mz - ny$ $Q = nx - lz$

And $R = ly - mx$

Therefore lagrange auxiliary equation will be

$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} \frac{dz}{ly-mx}$ -------- (1)

Taking l,m,n as multipliers

$\frac{ldx+mdy+ndz}{0} \Rightarrow ldx + mdy + ndz = 0$

On integration $lx + my + nz = c_1$ --------(2)

Again x, y, z as multipliers

$\frac{xdx+ydy+zdz}{0} \Rightarrow xdx + ydy + zdz = 0$

On integration $x^2 + y^2 + z^2 = c_2$

Therefore general solution of given equation will be

$\phi\,(lx + my + nz, x^2 + y^2 + z^2) = 0$

Multiple Choice Questions

Q.1 The order of the following partial differential equation $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 m^2$ is

(A) 2 (B) 1 (C) 3 (D) None of these

Answer (B)

Q.2 A first order Partial differential equation in two independent variable x and y is given by

(A) $f(x, y, z, p, q) \neq o$

(B) $f(x, y, z, p) = o$

(C) $f(x, y, z, p, q) = o$

(D) None of these

Answer (C)

Q.3 The solution of following $p.d.e.\ 25r - 408 + 16t = 0$ is given by

(A) $\phi_1(5y + 4x) + x\phi_2(5y + 4x)$

(B) $\phi_1(4y + 5x) + \phi_2(5y + 4x)$

(C) $\phi_1(5y + 4x) + x\phi_2(5y - 4x)$

(D) None of these

Answer (A)

Q.4 For the factor $(m + 1)^2 (m - 1)^2 = 0$ here the value of m =?

(A) $-1, -1, 1$ (B) $i, i, 1, 1$

(C) $-1, -1, i, i$ (D) None of these

Answer (A)

Q.5 The value of $\frac{1}{(D^2 + D')^2} x$ =?

(A) $\frac{x^5}{5}$ (B) $\frac{x^3}{6}$ (C) $\frac{x^6}{6}$ (D) $\frac{-x^3}{6}$

Answer (B) [2]

Q.6 The value of $\frac{1}{(2D - D')^2} \log(x + 2y)$ =?

(A) $\frac{x^2}{2!2^2} \log(x + 2y)$

(B) $\frac{x^2}{4} \log(x + 2y)$

(C) $\frac{x^1}{2!3^1}\ log\,(x+2y)$ (D) None of these

Answer (A)

Q.7 The value of $\frac{1}{(F(D,D')}\,e^{ax+by} = ?\ if\ f(a,b) \neq 0$

(A) $e^{ax+by}\frac{1}{F(D+a,\ D'+b)}$ (B) $\frac{1}{(F(a,b)}\,e^{ax+by}$

(C) $\frac{1}{(F(D-a)}\,e^{ax+by}$ (D) $\frac{1}{(F(a,b)}\,e^{ax-by}$

Answer (B)

Q.8 $\frac{3x^1+9}{x^3+9x} = ?$

(A) $log\ 3x^2+9$ (B) e^{3x^2+9}

(C) $log(x^3+9x)$ (D) $log(x^3+9)$

Answer (C)

Q.9 The order of the following differential Equation $\frac{d^1y}{dx^1}+11\frac{dy}{dx}+9y = sinx$ **is**

(A) 1 (B)3

(C) 2 (D) None of these

Answer (C)

Q.10 The general solution of the following differential equation $\frac{d^1y}{dx^1}+y = 0$ **is given by**

(A) $y = A\ cosx$ (B) $y = A\ cosx - B\ sinx$

(C) $y = A\sin x$ (D) $y = A\ sinx + B\cos x$

Answer (D)