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Concept based notes

Mechanics

Physics Paper I

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this endeavour. They played an active role in coordinating the various stages of this endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Dhirendra P Verma

Content

Unit 1

Physical Laws and Frame of Reference

Q.1 What is the inertial and non inertial frame of references? Explain clearly. Prove that the distance or displacement between two points is invariant under Galilean Transformation.

Ans **Frame of Reference:** Motion of a body is always described with reference to some well defined coordinate system. This coordinate system is referred to as 'frame of reference'. In three dimensional space a frame of reference consists of three mutually perpendicular lines called 'axes of frame of reference' meeting at a single point or origin.

Inertial Frame of Reference: A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called 'INERTIAL FRAME OF REFERENCE'. An inertial frame of reference is actually an un-accelerated frame of reference. Newton's laws of motion are valid in all inertial frames of reference. In this frame of reference a body does not acted upon by external forces.

Non-Inertial Frame of Reference: A frame of reference is said to be a non-inertial frame of reference when a body, not acted upon by an external force is accelerated. In non-inertial frame of reference, Newton's laws of motion are not valid.

Distance between two points is invariant under Galilean Transformation: Let us consider two inertial frame of reference S and S′. S′ is moving with a constant velocity v with respect to frame S. where $v = v_x i + v_y j + v_z k$. Again let the co-ordinates of two points in frame S is (x_1, y_1, z_1) and (x_2, y_2, z_2) and in frame S′ is (x'_1, y'_1, z'_1) and (x'_2, y'_2, z'_2) respectively.

From Galilean Transformation, we know that

$x' = x - v_x t,\ y' = y - v_y t,\ z' = z - v_z t$

Hence,

$x'_1 = x_1 - v_x t,\ y'_1 = y_1 - v_y t,\ z'_1 = z_1 - v_z t$

And

$x'_2 = x_2 - v_x t,\ y'_2 = y_2 - v_y t,\ z'_2 = z_2 - v_z t$

Now the distance between two point in frame S′ is =

$$\sqrt{\left\{(x'_2 - x'_1)^2 + \left(y'_2 - y'_1\right)^2 + (z'_2 - z'_1)^2\right\}}$$

On substituting the values of $x'_1, x'_2, y'_1, y'_2, z'_1$ and z'_2, we get the distance between two point in frame S′

=

$$\sqrt{\{\{(x_2 - v_x t) - (x_1 - v_x t)\}^2 + \{(y_2 - v_y t) - (y_1 - v_y t))\}^2 + \{(z_2 - v_z t) - (z_1 - v_z t)\}^2\}}$$

$= \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$

= the distance between two point in frame S

Hence we can say that distance between two points is invariant under Galilean Transformation.

Q.2 What are Galilean transformations? Prove that a reference frame moving with constant velocity with respect to an inertial frame is also an inertial frame.

Ans Consider two inertial frames S and S′. The frame S′ moves with a constant velocity v along the common x and x' axes, where $\vec{v}$ is measured relative to S. We assume that the origins of S and S′ coincide at t = 0 and that an event occurs at point P in space at some instant of time. An observer in S describes the event with space–time coordinates (x, y, z, t), whereas an observer in S′ uses the coordinates (x', y', z', t') to describe the same event. As we see from the geometry, the relationships among these various coordinates can be written

$x' = x - v_x t, \qquad y' = y - v_y t, \qquad z' = z - v_z t, \qquad t' = t$

These equations are the Galilean space–time transformation equations.

Let S′ is moving with a constant velocity with respect to frame S. Since velocity of frame S′ is constant, the acceleration of the S′ frame will be zero. So by the definition of inertial frames "An inertial frame of reference is actually an un-accelerated frame of reference. In this frame of reference a body does not acted upon by external forces.", the reference frame S′ moving with constant velocity with respect to an inertial frame is also an inertial frame.

Q.3 Write down Lorentz Transformation equations and explain Lorentz Fitzgerald contraction and time dilation.

Ans Let us consider two inertial frame of reference S and S′. S′ is moving with a constant velocity v with respect to frame S in x-direction. Then the Lorentz Transformation equations are given by:

$$x' = \frac{x-vt}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}},\ y'=y,\ z'=z,\ t' = \frac{t-\left({vx}/{c^2}\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$$

Lorentz Fitzgerald Contraction: The length of the object moving with constant velocity v relative to observer is contracted by a factor $\left[1-\frac{v^2}{c^2}\right]$ in the direction of motion while the length at right angles to the direction of motion remains unchanged. This phenomenon of length contraction is known as Lorentz Fitzgerald contraction.

Let us consider two frame of reference S and S′. S is an inertial frame of reference at rest and S′ is moving with a constant velocity with respect to frame S in +x-direction. Let a rod of proper length l_o is placed at rest in frame S′ along the x axis. If x'_1 and x'_2 are the co-ordinates of the ends of the rod in S′ frame, then

$l_o = x'_2 - x'_1$

If l is the length of the rod in frame S and x_1 and x_2 are the co-ordinates of the ends of the rod in S frame, then

$l = x_2 - x_1$

Now according to Lorentz Transformation, $x' = \frac{x-vt}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$, So, $x_1{}' = \frac{x_1 - vt}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$, $x_2{}' = \frac{x_2 - vt}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$

Thus, $l_o = x'_2 - x'_1 = \frac{x_2 - vt}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}} - \frac{x_1 - vt}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}, = \frac{x_2 - x_1}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}} = \frac{l}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$

$$l_o = \frac{l}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$$

This is the expression for the Lorentz Fitzgerald contraction.

Time Dilation: A clock will be found to run more and more slowly if the relative velocity between the clock and observer is increased more and more. This phenomenon is called Time Dilation.

Let us consider two frame of reference S and S′. S is an inertial frame of reference at rest and S′ is moving with a constant velocity with respect to frame S in +x-direction. Let the clock be situated in frame S and gives signals at interval Δt.
Where $\Delta t = t_2 - t_1$

Then the interval observed by observer in frame S′ will be $\Delta t' = t'_2 - t'_1$

Now according to Lorentz Transformation $t' = \frac{t - \left({vx}/{c^2}\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$

So, $t_1{}' = \frac{t_1 - \left({vx}/{c^2}\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$ and $t_2{}' = \frac{t_2 - \left({vx}/{c^2}\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$

Hence $\Delta t' = t_2{}' - t_1{}' = \frac{t_2 - \left({vx}/{c^2}\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}} - \frac{t_1 - \left({vx}/{c^2}\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}} = \frac{t_2 - t_1}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}} = \frac{\Delta t}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$

$$\Delta t' = \frac{\Delta t}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$$

This is the expression for Time Dilation.

Q.4 Average value of μ-meson at rest is 2.2X 10$^{(-6)}$ second. What will be the average age of μ -meson if it enters earth's atmosphere with the velocity of 0.98c? What will be the average distance covered by it before decay?

Ans According to expression for time dilation formula

$$\Delta t^{'} = \frac{\Delta t}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$$

Here $\Delta t = 2.2\text{X } 10(-6)$

$$\Delta t^{'} = \frac{2.2\text{X } 10(-6)}{\sqrt{\left[1 - \frac{(0.98c)^2}{c^2}\right]}}$$

$$\Delta t^{'} = \frac{2.2\text{X } 10(-6)}{\sqrt{[1 - (0.98)^2]}}$$

$$\Delta t^{'} = \frac{2.2\text{X } 10(-6)}{0.19899}$$

$$\Delta t^{'} = 11.056\ X10^{-6}\ sec$$

This is the average age of μ -meson if it enters earth's atmosphere with the velocity of 0.98c.

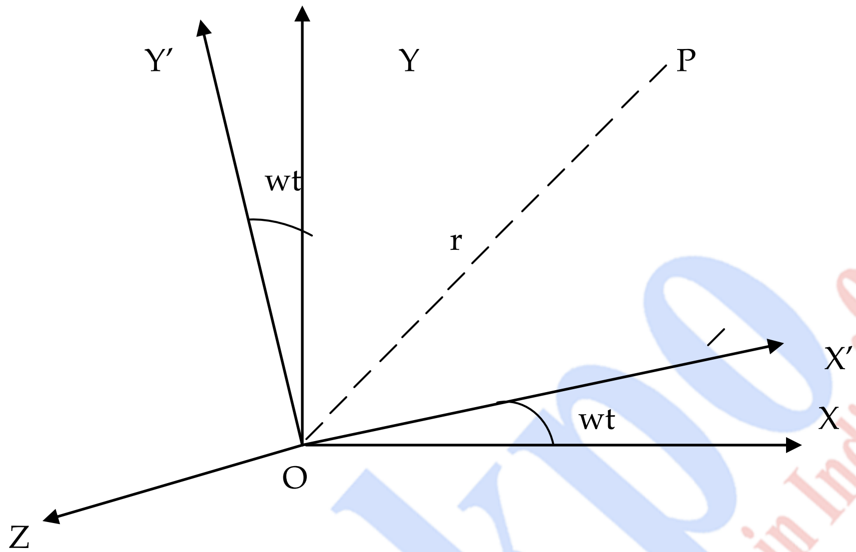
Now the average distance covered by it before decay = $11.056\ X10^{-6} X\ 0.98c$

$$= 11.056\ X10^{-6} X\ 0.98\ X\ 3\ X\ 10^8\ m$$

$$= 32.5041\ X\ 10^2\ m = \ 3.250\ X\ 10^3 m = \ 3.250\ Km$$

Q.5 The origin of two frames of reference S and S′ are at the same point and ZZ′ axes coincide. The frame S′ is rotating with a uniform angular velocity along Z axis. Deduce transformation equations for Cartesian coordinates of these frames.

Ans Let us consider two frame of refrences S and S′ where S′ is moving with an angular velocity ω relative to frame S about Z- axis. Let the origin and axes of two frames coincide initially, i.e. at t= t′= 0. After time t, X′ and Y′ are rotated by an angel ωt relative to x and y axes respectivily.



The observation taken by Observer in S′ is (x′,y′,z′) and the observation taken by Observer in S is (x,y,z) at a point P. The relation between (x′,y′,z′) and (x,y,z) is given by:

x′ = component of x along x-axis + component of y along x-axis + component of z along x-axis

$x' = x\cos\omega t + y\sin\omega t + z\cos 90°$

$x' = x\cos\omega t + y\sin\omega t$

Similarly,

y′ = component of x along y-axis + component of y along y-axis + component of z along y-axis

$y' = -x\sin\omega t + y\cos\omega t + z\cos 90°$

$y' = -x\sin\omega t + y\cos\omega t$

and

z′ = component of x along z-axis + component of y along z-axis + component of z along z-axis

$z' = x\cos 90° + y\sin 90° + z\cos 0°$

z′ = z

and t′ = t

Hence

The transformation equations for Cartesian coordinates of given frame is:

x′ = xcosωt + ysinωt

y′ = -xsinωt + ycosωt

z′ = z

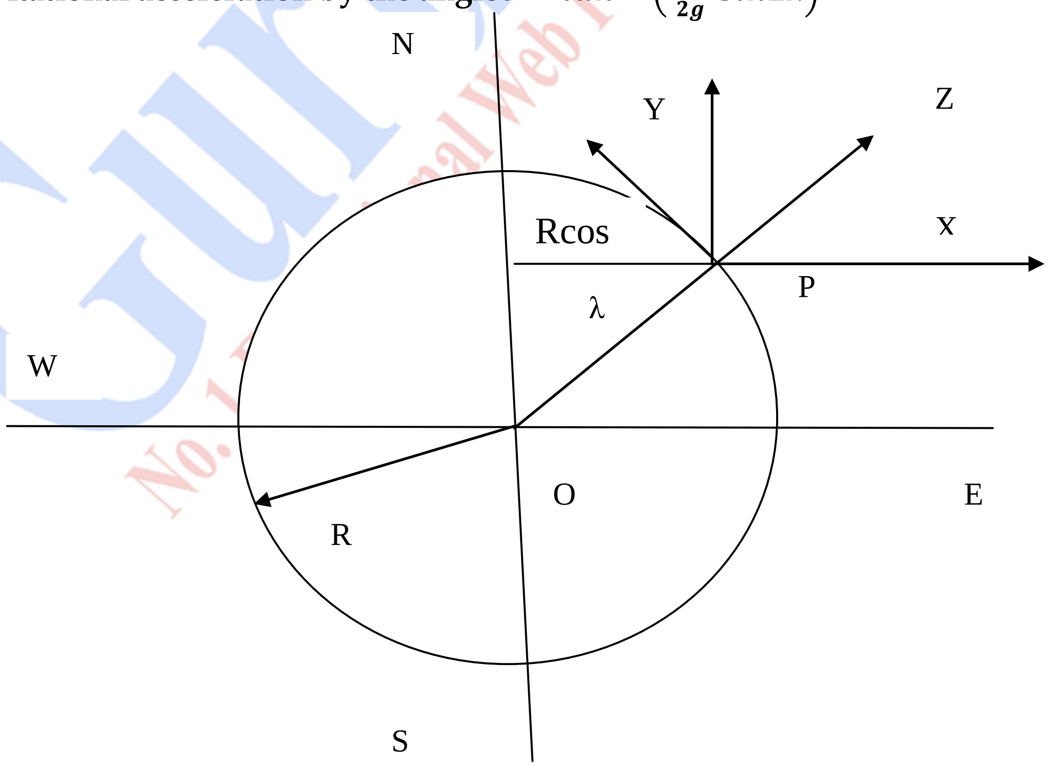
t′ = t

Q.6 Prove that at latitude λ the acceleration due to gravity is related to its real value g by the relation-

$$|\vec{g_\lambda}| = \sqrt{(g - \omega^2 R\cos^2\lambda)^2 + \omega^4 R^2 \cos^2\lambda \sin^2\lambda}$$

Derive the value of $\vec{g_\lambda}$ at poles and equator of earth.

Prove that due to rotation of the earth the direction of observed gravitational acceleration g_λ at λ- latitude is inclined to g, the real gravitational acceleration by the angle $\theta = \tan^{-1}\left(\frac{\omega^2 R}{2g}\sin 2\lambda\right)$



Let us consider a particle P at a latitude λ on the earth's surface. If the radius of the earth is R, the particle will be moving in a circular path of radius Rcos λ due to rotation of earth with an angular velocity ω. Let the horizontal line through P towards east be taken as X-axis, along north as Y-axis and Z-axis points vertically up. Then we have

$$\vec{\omega} = \omega\, cos\lambda\, \hat{j} + \omega\, sin\lambda\, \hat{k}$$

And the position vector of point P is

$$\vec{r} = R\hat{k}$$

The resultant force acting on the particle of mass m at P is given by

$$F' = F - 2m\vec{\omega} X \dot{\vec{r}} - m\vec{\omega} X (\vec{\omega} X \vec{r})$$

Where F′ is the force in rotating frame of earth and F is the real force in inertial frame. If g is the true value of acceleration due to gravity if earth were not rotating and $\boldsymbol{g_\lambda}$ is the apparent acceleration due to gravity when earth is rotating, then we have

$$m\vec{g_\lambda} = -mg\hat{k} - 2m\vec{\omega} X \dot{\vec{r}} - m\vec{\omega} X (\vec{\omega} X \vec{r})$$

Since the particle is assumed to be at rest i.e. $\dot{\vec{r}} = 0$, no coriolis force acts on the particle. Hence

$\vec{g}_\lambda = -gk^\lambda - \vec{\omega} x (\vec{\omega}\ X\ \vec{y}\ )$

But

$\overrightarrow{\omega X \vec{r}} = (\omega \cos\lambda + \omega \sin\lambda\, \hat{k})\, x\, R\ \hat{k}\ = \omega R\, cos\lambda\, \hat{r})$

And

$\vec{\omega}\ X\ (\vec{\omega} x \vec{r}) = (\omega \cos\lambda\, \hat{j} + \omega \sin\ \hat{k})\ X\ \ \omega R\, cos\lambda\, \hat{r})$

$\vec{\omega}\ X\ (\vec{\omega} x \vec{r}) = \omega^2 R\, cos^2\lambda\, \hat{k} + X\, \omega R \cos\lambda \sin\lambda\ \hat{j}$

Or

Hence we have

$\vec{g_\lambda}$ =- $g\hat{k} - (-\omega^2 R \cos^2\lambda \hat{k} + \omega^2 R\ \cos\ \ \lambda \sin\ \hat{j})$

Or

$\vec{g_\lambda} = -(g - \omega^2 R\, cos^2\lambda)\hat{k} - \omega^2 R\, cos\lambda sin\lambda\, \hat{j}$

Hence the magnitude of apparent value of g is given by

$$\overrightarrow{|g_\lambda|} = \sqrt{(g - \omega^2 R\, cos^2\lambda)^{\,2} + \omega^4 R^2\, cos^2\lambda\, sin^2\lambda}$$

$$|\vec{\,}| = \sqrt{(- {}^{22})^2 + {}^{422\ 2}}$$

And If the direction of g at P makes an angel θ with the Z-axis, then

$\theta = tan^{-1}\left(\frac{\omega^2 R}{2g} sin\ 2\lambda\right)$= -1$\left(\frac{2}{2}2\right)$

Q.7 Find the effective weight of a person whose mass is 60 kg. He is going vertically upward in a rocket with acceleration 6g.

Ans Since the person is going vertically upward in a rocket with acceleration 6g, then a pseudo force will act on the person vertically downward. Since the person has its mass 60 kg, so total effective weight of the person is = 60X g + 60X6g= 60 X 7g N = 420g N

Q.8 Find our earth's gravitational acceleration at 60° latitude, if rotation of earth is stopped, where radius of earth is 6.37X 10^6 meter.

Ans At latitude λ the acceleration due to gravity is related to its real value g by the relation-

$$\overrightarrow{|g_\lambda|} = \sqrt{(g - \omega^2 R\, cos^2\lambda)^{\,2} + \omega^4 R^2\, cos^2\lambda\, sin^2\lambda}$$

Since λ = 60°, and ω = 0, then

$$\overrightarrow{|g_\lambda|} = \sqrt{\left(g - (0)^2 R\, cos^2(60^0)\right)^{\,2} + 0^4 R^4 cos^2 60\ \ sin^2 60}$$

$$|\vec{\,}| = \sqrt{(g - 0)^2) \ \ + 0^4 R^4 cos^2 60\ \ sin^2 60}$$

$\overrightarrow{|g_\lambda|}$=$g$

Q.9 In how much time will the oscillation plane of a Foucault pendulum complete one turn when it is placed:-

1. **At equator**
2. **At 45° North latitude**
3. **At north Pole**

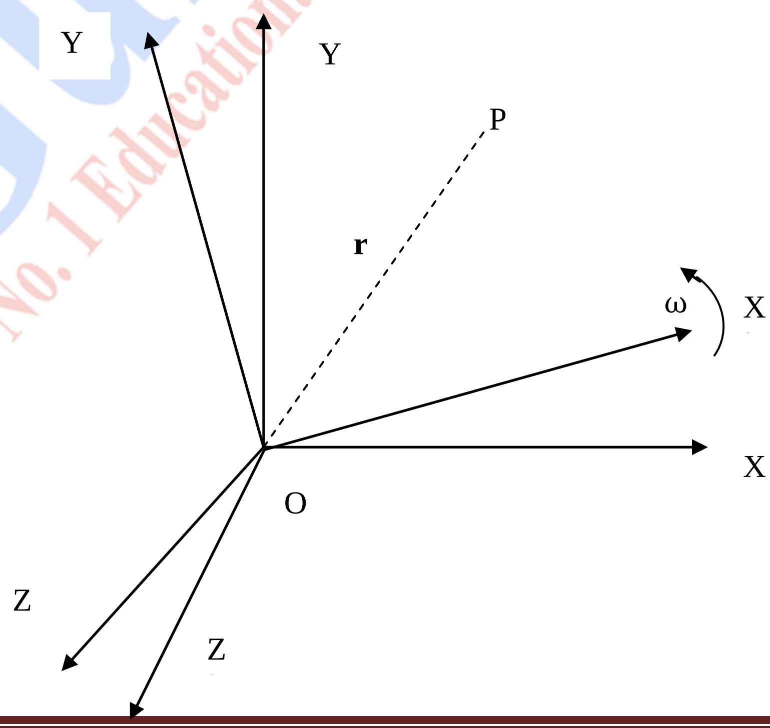
Ans At any latitude λ, the time of one rotation of the plane of oscillation of Foucault pendulum is given by:

$$TT = \frac{24}{\sin\lambda} Hours$$

1. At equator, $\lambda = 0^0$, T = Infinite
2. At 45° North latitude, $\lambda = 45^0$, T = 24 sqrt (2) = 24 X 1.44 = 33.94 Hr
3. At north Pole, $\lambda = 90^0$, T = 24 Hr

Q.10 Define coriolis force. Derive an expression for coriolis force acting on a moving mass m in rotating frame of reference. Find the effect of coriolis force on a freely falling body.

Ans The **Coriolis effect** is a deflection of moving objects when they are viewed in a rotating reference frame. In a reference frame with clockwise rotation, the deflection is to the left of the motion of the object; in one with counter-clockwise rotation, the deflection is to the right. Newton's laws of motion describe the motion of an object in a (non-accelerating) inertial frame of reference. When Newton's laws are transformed to a uniformly rotating frame of reference, the Coriolis and centrifugal forces appear. Both forces are proportional to the mass of the object. The Coriolis force is proportional to the rotation rate and the centrifugal force is proportional to its square. The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame and is proportional to the object's speed in the rotating frame. The centrifugal force acts outwards in the radial direction and is proportional to the distance of the body from the axis of the rotating frame. These additional forces are termed inertial forces, fictitious forces or *pseudo forces*. They allow the application of Newton's laws to a rotating system. They are correction factors that do not exist in a non-accelerating or inertial reference frame.



Consider two system S and S′ where S is at rest and S′ is in uniform rotation relative to frame S. Let the origin of both the system coincide. Let **i,j,k** be the unit vectors associated with X, Y, Z of the system and **i′, j′,k′** the unit vectors associated with axes X′ Y′ Z′ of system S′. Then the position vector r in terms of its components along the axes can be expressed as

$$r = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x\boldsymbol{i'} + y\boldsymbol{j'} + z\boldsymbol{k'}$$

An observer in frame S′ situated at origin will observe the time derivative of **r** to be

$$\frac{dr}{dt} = \frac{dx'}{dt}\boldsymbol{i'} + \frac{dy'}{dt}\boldsymbol{j'} + \frac{dz'}{dt}\boldsymbol{k'}$$

But the time derivative of **r** relative to an observer in system S will be

$$\frac{dr}{dt} = \frac{dx'}{dt}\boldsymbol{i}' + \frac{dy'}{dt}\boldsymbol{j}' + \frac{dz'}{dt}\boldsymbol{k}' + x'\frac{di'}{dt} + y'\frac{dj'}{dt} + z'\frac{dk'}{dt}$$

Since the unit vectors i′, j′,k′ change with time relative to the observer fixed in system x. These unit vectors are rotating with uniform angular velocity ω, we have

$$\frac{di'}{dt} = \vec{\omega}\, X\, \boldsymbol{i'},\ \frac{dj'}{dt} = \vec{\omega}\, X\, \boldsymbol{j'},\ \frac{dj'}{dt} = \vec{\omega}\, X\, \boldsymbol{k'}$$

Also $\frac{d'r}{dt} = \frac{dx'}{dt}\boldsymbol{i'} + \frac{dy'}{dt}\boldsymbol{j'} + \frac{dz'}{dt}\boldsymbol{k'}$

Where the differential referred to S′ is denoted by $\frac{d'}{dt}$

Hence from above equations, we have

$$\frac{d\boldsymbol{r}}{dt} = \frac{d'\boldsymbol{r}}{dt} + x'(\vec{\omega}\, X\, \boldsymbol{i}') + y'(\vec{\omega}\, X\, \boldsymbol{j}') + z'(\vec{\omega}\, X\, \boldsymbol{k}')$$

$$\frac{d\boldsymbol{r}}{dt} = \frac{d'\boldsymbol{r}}{dt} + \vec{\omega}\, X(x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'^{\boldsymbol{k}'})$$

i.e.

$$\frac{d\boldsymbol{r}}{dt} = \frac{d'\boldsymbol{r}}{dt} + \vec{\omega}\, X\, \boldsymbol{r}$$

or the operator equation from above equation can be written as

$$\frac{d}{dt} = \frac{d'}{dt} + \vec{\omega}\, X$$

Again differentiating the above equation, we get

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt}\right)$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = \left(\frac{d'}{dt} + \vec{\omega}\, X\right)\left(\frac{d'\boldsymbol{r}}{dt} + \vec{\omega}\, X\, \boldsymbol{r}\right)$$

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d'^2\boldsymbol{r}}{dt^2} + 2\left(\vec{\omega}\, X\, \frac{d'\boldsymbol{r}}{dt}\right) + \vec{\omega}\, X(\vec{\omega}\, X\, \boldsymbol{r}) + \frac{d'\omega}{dt}\, X\, \boldsymbol{r}$$

The above formula is called **Coriolis' theorem**. In this theorem the term $\frac{d^2\boldsymbol{r}}{dt^2}$ represents the acceleration of the particle relative to system S, the term $\frac{d^2\boldsymbol{r}'}{dt^2}$ represents the acceleration of the particle relative to system S′, the term $2\left(\vec{\omega}\, X\, \frac{d'\boldsymbol{r}}{dt}\right)$ only appears when the particle moves in system S′ and is known as Coriolis' acceleration, the term $\vec{\omega}\, X(\vec{\omega}\, X\, \boldsymbol{r})$ is the centripetal acceleration of the point in rotation about an axis and the term $\frac{d'\omega}{dt}\, X\, \boldsymbol{r}$ vanishes for constant angular velocity of rotation about a fixed axis.
The equation in terms of force can be written as:

Force acting on the particle of mass m in frame S is given by

$$F = m\frac{d^2\boldsymbol{r}}{dt^2} = m\frac{d^2\boldsymbol{r}}{dt^2} + 2\, m\left(\vec{\omega}\, X\, \frac{d'\boldsymbol{r}}{dt}\right) + m\vec{\omega}\, X(\vec{\omega}\, X\, \boldsymbol{r}) + m\frac{d'\omega}{dt}\, X\, \boldsymbol{r}$$

And the Force acting on the particle of mass m in frame S′ is given by

$$F' = \frac{md^2}{dt^2} = \frac{md^2\boldsymbol{r}}{dt^2} + 2\, m\left(\vec{\omega}\, X\, \frac{d'\boldsymbol{r}}{dt}\right) + m\vec{\omega}\, X(\vec{r}\, X\, \omega) - \frac{md'\omega}{dt}\, X\, \boldsymbol{r}$$

′Or

$$F' = F - 2\, m\left(\vec{\omega}\, X\, \frac{d'\boldsymbol{r}}{dt}\right) - m\vec{\omega}\, X(\vec{\omega}\, X\, r) - \boldsymbol{m}\,\frac{d'\omega}{dt}\, X\, \boldsymbol{r}$$

Here, the term $-m\vec{\omega}\boldsymbol{x}(\vec{\omega}\, X\, \vec{r})$ is called the centrifugal force.

The term $-2m\left(\vec{\omega}\, X\, \frac{d'r}{dt}\right)$ represent the fictious force acting on the particle in the non-inertial frame and is known as Coriolis force.
The term $-2m\frac{d'\omega}{dt}$ x r appears only in case of non-uniform rotation.

Effect of coriolis force on a freely falling body:

The earth is moving an angular velocity $\vec{}$ with north south as the axis of rotation. Let X′, Y′, Z′ be the axes along the east, north and vertical directions respectively. And λ,j′,k′ be the unit vectors along X′, Y′ and Z′ respectively.

Let the body of mass m be falling freely on the surface of the earth. Let v′ be the velocity of the body at any instant t.

Then, $\omega^{\rightarrow} = \omega\cos\lambda' + \omega\sin\lambda'$

And **v′** = -v′k′

Where v′ is the magnitude of velocity **v′**. Negative sign represents that the body is moving vertically downward, while positive Z axis is vertically upward.

The coriolis force is given by

$= F_C = -2m(\vec{\omega} X v')$

$= 2m[(\omega\cos\lambda\, j' + \omega\sin\lambda k') X (-v' - k')]$

$= 2m\, v\, \omega\cos\lambda i'$

This equation represent that the coriolis force, is acting along the + x axis.

Therefore the deflection of the body will be towards east in northern hemisphere.Hence the acceleration along the x-axis is given by

$\frac{d^2x}{dt^2} = 2\, v\, \omega\, \cos\lambda$

If g is the acceleration due to gravity, then

V= u + at,

Here u=0 and a = g

Then v = gt

Hence $\Rightarrow \frac{d^2x}{dt^2} = 2gt\, \omega\cos\lambda$

On integrating this euation w.r.to t, we get

$\Rightarrow \frac{d\, r}{dt} = g\omega t\, \cos\lambda + A$

Where a is an arbitrary constant whose value can be obtained by initial condition i.e. at t=0, $\frac{d\, x}{dt} = 0$

This gives A= 0

Hence $\frac{dx}{dt} = g\omega\, t^2\cos\lambda$

Again integrating the above equation, we get

$$x = g\omega \frac{t^3}{3} \cos\lambda + B$$

Where B is another arbitrary constant whose value can be obtained by initial condition i.e. initially the deviation is zero, i.e. at t=0, x=0

This gives B=0

Hence $x = \frac{1}{3}\, g\omega\, t^3 \cos\lambda$

This equation represents the deviation along x-axis at any time t at latitude λ.

If h is the hight traversed by the body in time t, then we have

$h = \frac{1}{2} g t^2$

or $t = \sqrt{\frac{2h}{g}}$

on substituting the value of t in above equation, we have

$$x = \frac{1}{3}\, g\omega \left(\sqrt{\frac{2h}{g}}\right)^3 \cos\lambda$$

Or $x = \frac{2}{3}\, \omega h \cos\lambda \sqrt{\frac{2h}{g}}$

This is the deviation of the falling body from the vertical due to the rotation of earth towards east in the northern hemisphere and towards south in the southern hemisphere.

Q.8 Calculate velocity of a rod along its length, by which it is to be moved so that its length contracts by 50%.

Ans Let L be the length of the rod at rest, and when it moves along its length with velocity v, then its length after contracting becomes 0.5L

The length contraction formula is given by:

$l_o = \frac{l}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$

So l_o = L, l= 0.5L

$$L = \frac{0.5\, L}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$$

$$\sqrt{\left[1-\frac{v^2}{c^2}\right]} = 0.5$$

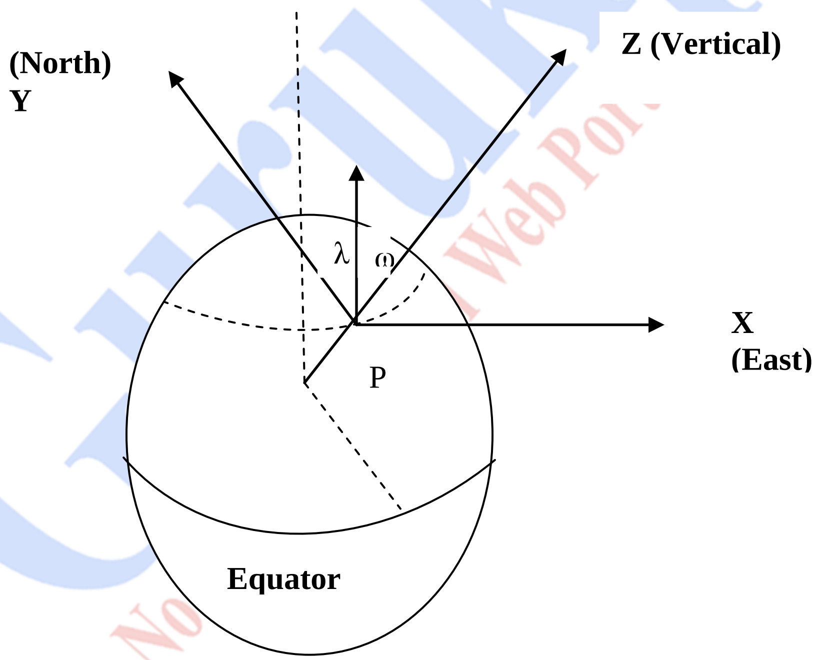
$$\left[1-\frac{v^2}{c^2}\right] = 0.25$$

$$\frac{v^2}{c^2} = 0.75$$

$$v = \sqrt{0.75}\, c$$

Q.9 A body is throne vertically upward with velocity u. Prove that it will fall at a point in west direction from its initial position where displacement is

$$\left(\frac{8}{3} whcos\lambda\right)\left(\frac{2h}{g}\right)^{1/2}$$



Ans The angular velocity of rotation of earth is given as:

$$\vec{\omega} = \omega\, cos\lambda\, \hat{j} + \omega\, sin\lambda\, \hat{k}$$

Since the the particle at point P has to attain a height h, its velocity of projection is given by:

$$\vec{v} = v_0\,\hat{k} = \sqrt{2gh}\,\hat{k}$$

And its velocity at any time t is given by

$$\vec{v'} = (v_0 - gt)\,\hat{k}$$

The only effective fictious acceleration in this case is coriolis acceleration, given by $-2\,\vec{\omega}\,X\,\vec{v'}$. the centrifugal acceleration is very small so it can be neglected

Hence coriolis acceleration is given by:

$$a' = -2\,\vec{\omega}\,X\,\vec{v'} = -2\,\vec{\omega}\,X\,\vec{v'} = -2\left(\omega\,cos\lambda\,\hat{j} + \omega\,sin\lambda\,\hat{k}\right)X(v_0 - gt)\hat{k}$$

Or

$$a' = -2\omega\,(v_0 - gt)cos\lambda\,\hat{\imath}$$

i.e. the particle appears to have an acceleration towards west as we have taken x-axis towards east where as the acceleration due to gravity is in y-direction. If at any time t the displacement due to this coriolis force is x, then magnitude of acceleration is given by:

$$\frac{d^2x}{dt^2} = 2\omega\,(v_0 - gt)cos\lambda$$

Integrating above equation with respect to t, we get

$$\frac{dx}{dt} = 2\omega\left(v_0 t - g\frac{t^2}{2}\right)cos\lambda + A$$

Where A is an arbitrary constant and can be obtained by initial condition that at t=0, $\frac{dx}{dt} = 0$, hence we get A = 0. Thus we get

$$\frac{dx}{dt} = 2\omega\left(v_0 t - g\frac{t^2}{2}\right)cos\lambda$$

On Integrating above equation with respect to t, we get

$$x = 2\omega\left(v_0\frac{t^2}{2} - g\frac{t^3}{6}\right)cos\lambda + B$$

Where A is an arbitrary constant and can be obtained by initial condition that at t=0, x=0

This gives B= 0

Hence

$$x = 2\omega\left(v_0\frac{t^2}{2} - g\frac{t^3}{6}\right)cos\lambda$$

Or

$$x = \omega\left(v_0 - g\frac{t}{3}\right)t^2 cos\lambda$$

This is the general expression for the horizontal x-displacement of the particle after tme t of its projection.

If t_0 is the time for its return journey,then

$$t_0 = 2\frac{v_0}{g} = \frac{2}{g}\sqrt{2gh} = 2\sqrt{\frac{2h}{g}}$$

Put this value of t in the above general equation for horizontal displacement, we get the net horizontal displacement of particle from the point of projection on its return is:

$$x_0 = \omega\left(\sqrt{2gh} - g\frac{2\sqrt{\frac{2h}{g}}}{3}\right)\left(2\sqrt{\frac{2h}{g}}\right)^2 cos\lambda$$

$$x_0 = \omega\left(\sqrt{2gh} - \frac{2\sqrt{2gh}}{3}\right)\frac{8h}{g}cos\lambda$$

$$x_0 = \omega\left(\frac{\sqrt{2gh}}{3}\right)\frac{8h}{g}cos\lambda$$

$$x_0 = \frac{8}{3}\omega\, cos\lambda\sqrt{\frac{2h^3}{g}}$$

Q.10 With how much velocity should a clock move so that it appears one minute slow in a day?

Ans One day = 24 hours = 24X60 min = 1440 min
If the clock moves with a velocity such that it appears one minute slow, then clock must record 1439 minute recorded by clock with respect to the observer in stationary frame.

Hence according to Einstien's formula for apparent retardation of clocks, we must have

$$\Delta t' = \frac{\Delta t}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$$

$$1440 = \frac{1439}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$$

$$\sqrt{\left[1 - \frac{v^2}{c^2}\right]} = \frac{1439}{1440}$$

$$\frac{v^2}{c^2} = 1 - \frac{(1439)^2}{(1440)^2}$$

$$\frac{v^2}{c^2} = \frac{(1440)^2 - (1439)^2}{(1440)^2}$$

$$\frac{v^2}{c^2} = \frac{2879}{(1440)^2}$$

$$v = \frac{\sqrt{2879}}{1440}\, c$$

Q.11 Discuss the postulates of special theory of relativity. Derive Lorentz transformation equation.

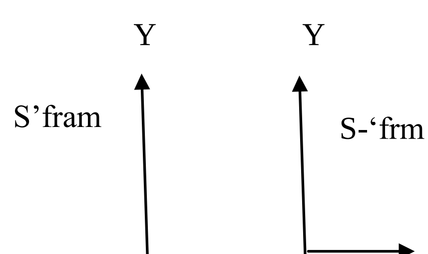
Ans **Postulates of special theory of relativity:** The absence of any fringe shift in the Michelson-Morely experiment for any orientation of the interferometer and for any time of the year negated the ether hypothesis for light propagation. Light waves are oscillations of the electromagnetic field, and no propagation medium is necessary, unlike sound waves. However, if Galilean transformations are correct, then Maxwell's equations must be modified for every possible reference frame to account for different velocities for the speed of light. Einstein assumed the opposite: that Maxwell's equations are fundamentally correct, but that our intuitive Galilean transformation is not.

This led to the following two postulates:

1. The laws of physics, including electromagnetism, are the same in all inertial frames.
2. Every observer measures the same value c for the speed of light (in vacuum) in all inertial frames.

The second postulate is really a consequence of the first, because if Maxwell's equations hold in all inertial frames, then the only possible value for the speed of light is c.

Lorentz Transformation Equation: Let us consider two inertial frame of reference S and S′. S is an inertial frame of reference at rest and S′ is moving with a constant velocity with respect to frame S in +x-direction.



Now suppose that there is a single flash at the origin of S and S′ at time $t = t' = 0$, when the two inertial frames happen to coincide. The outgoing light wave will be spherical in shape moving outward with a velocity c in *both* S and S′ by Einstein's Second Postulate.

$x^2 + y^2 + z^2 = c^2t^2$

$x'^2 + y'^2 + z'^2 = c^2t'^2$

or $x^2 + y^2 + z^2 - c^2t^2 = 0$

$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$

From the above two equations, we get

$x^2 + y^2 + z^2 - c^2t^2 = \lambda\,(x'^2 + y'^2 + z'^2 - c^2t'^2)$

As the velocity of S′ is only along x-axis, thus

$y' = y$

$z' = z$

Which gives $\lambda = 1$,

So, $x^2 - c^2t^2 = x'^2 - c^2t'^2$

Now for the transformation equation relating to x and x′ is given by

$$x' = \gamma(x - vt)$$

Where γ is a constant.

Since motion is relative, we can assume that S is moving with respect to S′ with a velocity (-v) in –x-direction, then from the above transformation

$$x = \gamma(x' - vt')$$

From above both the transformation we have

$$x = \gamma\{\gamma(x - vt) - vt'\}$$

Or $$t' = \gamma\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right]$$

Now put the value of x′ and t′ in equation $x^2 - c^2t^2 = x'^2 - c^2t'^2$

$$x^2 - c^2t^2 = \gamma^2(x - vt)^2 - c^2\gamma^2\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right]^2$$

On solving this equation, we get the following quadratic equation

$$x^2 - c^2t^2 - \gamma^2(x^2 - 2vxt + v^2t^2) + c^2\gamma^2\left\{t^2 - \frac{2xt}{v}\left(1 - \frac{1}{\gamma^2}\right) + \frac{x^2}{v^2}\left(1 - \frac{1}{\gamma^2}\right)^2\right\} = 0$$

Since this equation is an identity, the coefficient of various powers of x and t must vanish separately.

Now equating the coefficient of xt to zero, we have

$$2v\gamma^2 + c^2\gamma^2\left\{\frac{-2}{v}\left(1 - \frac{1}{\gamma^2}\right)\right\} = 0$$

Which gives $\gamma = \frac{1}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$

Now we have the transformation equation as

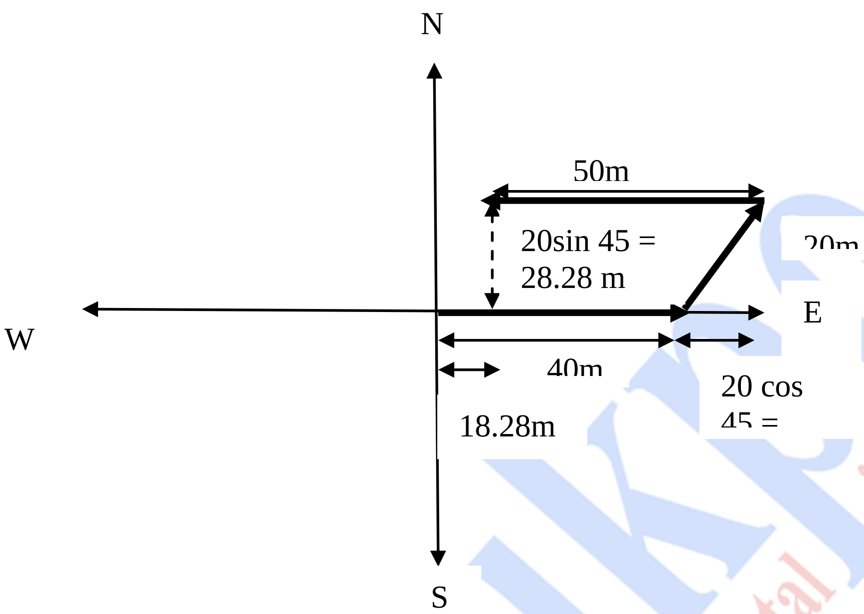
$x' = \frac{x - vt}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$, $y'=y$, $z'=z$,

And $t' = \gamma\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right]$ which gives $t' = \frac{t - \left({vx}/{c^2}\right)}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$

Hence the Lorentz transformation equations are

$x' = \frac{x - vt}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$, $y'=y$, $z'=z$ $t' = \frac{t - \left({vx}/{c^2}\right)}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}}$,

Q.12 A body is moving with a uniform velocity of 60 m/hr in east direction for 40 min. Then for 20 minutes it moves in a direction making an angle of 45° with north towards east and in the end moves for 50 minutes in west. What is the average velocity throughout the motion?



Since body is moving with a uniform velocity of 60 m/hr (60 m/hr = 1 m/min) in east direction for 40 min, so the distance travelled by the body in east = 40m.
Then 20 minutes it moves in a direction making an angle of 45° with north towards east, so the distance travelled by the body in north-east = 20 m
Again it moves for 50 minutes in west, so the distance travelled by the body in west direction = 50 m

The average velocity throughout the motion = Displacement/ Total time taken

$$= \frac{\sqrt{(28.28)^2+(18.28)^2}}{60+20+50}\frac{m}{min} = \frac{799.7524+334.1584}{130}\frac{m}{min} = \frac{799.7524+334.1584}{130}\frac{m}{min} =$$

$$\frac{1133.9108}{130}\frac{m}{min} = 8.722\frac{m}{min}$$

Q.13 Calculate the acceleration of a frame of reference, fixed to the earth at the equator and rotating with the earth once a day. If it is assumed to be an inertial frame , what will be the error in one minute in the recorded

position of particle near the surface of the earth? (Radius of earth= 6.4X10⁶m)

Ans If a frame of reference, fixed to the earth at the equator and rotating with the earth once a day, centripetal acceleration of frame is given by

$$Centripital\ force = \frac{v^2}{R} = \frac{(\omega R)^2}{R} = \omega^2 R$$

The angle travelled by earth in one day = 2**π** radians

And One day = 24 hours = 24X60X 60 sec = 86400 sec

The angular velocity of earth $\omega = \frac{Angle}{Time} = \frac{2\pi}{86400} = 0.73\ X\ 10^{-4}\ per\ sec$

Radius of earth= $6.4X10^6$m

So, the centripetal acceleration of frame fixed to the earth's surface is given by

$$= \omega^2 R = (0.73\ X\ 10^{-4})^2 6.4\ \text{X}\ 10^6 = 3.4\ X\ 10^{-2}\ m/s^2$$

If this frame is assumed to be an inertial frame, then

Q.15 Obtain the relativistic law of addition of velocities. Prove that the speed of a particle moving with speed of light is same in all inertial reference frames.

Ans **Relativistic law of addition of velocities:** Let us consider two frame of reference S and S′. S is an inertial frame of reference at rest and S′ is moving with a constant velocity with respect to frame S in +x-direction.Let a particle is mving with velocity u and u' in frame S and S' respectively, then

$$u = u_x\vec{i} + u_y\vec{j} + u_z\vec{k}, u' = u'_x\vec{i} + u'_y\vec{j} + u'_z\vec{k}$$

Where $u_x, u_y\ and\ u_z$ are components of u along x, y and z axes respectively

and $u'_x, u'_y\ and\ u'_z$ are components of u′ along x′, y′ and z′ axes respectively.

Now we have $u_x = \frac{dx}{dt}, u_y = \frac{dy}{dt}, u_z = \frac{dz}{dt}$

and $u'_x = \frac{dx\prime}{dt\prime}, u'_y = \frac{dy\prime}{dt\prime}, u'_z = \frac{dz\prime}{dt\prime}$

Now according to Inverse Lorentz Transformation

$$x = \frac{x' + vt'}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}, y = y', z = z', t = \frac{t' + \left(vx'/c^2\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$$

On differentiating these equations, we get

$$dx = \frac{dx' + vdt'}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}, dy = dy', dz = dz', dt = \frac{dt' + \left(vdx'/c^2\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}$$

Now from these equations we have,

$$u_x = \frac{dx}{dt} = \frac{\frac{dx' + vdt'}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}}{\frac{dt' + \left(vdx'/c^2\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}} = \frac{dx' + vdt'}{dt' + \left(vdx'/c^2\right)} = \frac{\frac{dx'}{dt'} + v}{1 + \left(v\frac{dx'}{dt'}/c^2\right)} = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\frac{dt' + \left(vdx'/c^2\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}} = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}dy'}{dt' + \left(vdx'/c^2\right)} = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}\frac{dy'}{dt'}}{1 + \left(v\frac{dx'}{dt'}/c^2\right)} = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}u'_y}{1 + \frac{v}{c^2}u'_x}$$

and $u_z = \frac{dz}{dt} = \frac{dz'}{\frac{dt' + \left(vdx'/c^2\right)}{\sqrt{\left[1-\frac{v^2}{c^2}\right]}}} = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}dz'}{dt' + \left(vdx'/c^2\right)} = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}\frac{dz'}{dt'}}{1 + \left(v\frac{dx'}{dt'}/c^2\right)} = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}u'_z}{1 + \frac{v}{c^2}u'_x}$

These equations represent transformation of velocity components from one inertial frame to another.

Speed of a particle moving with speed of light is same in all inertial reference frames:

Let us consider a particle is moving with speed of light c in S′ inertial frame of reference which is moving with a constant velocity v along x-direction, then $u'_x = c, u'_y = 0$ and $u'_z = 0$, so that $u' = c$

Now if we calculate the velocity component in frame of reference S which is stationary, then according to Relativistic law of addition of velocities we have,

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x} = \frac{c + v}{1 + \frac{v}{c^2}c} = \text{c}$$

$$u_y = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}(0)}{1+\frac{v}{c^2}c} = 0$$

and $u_z = \frac{\sqrt{\left[1-\frac{v^2}{c^2}\right]}(0)}{1+\frac{v}{c^2}c} = 0$

Hence the speed of the particle in S frame is $u = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2} =$

$\sqrt{c^2 + 0 + 0} = c$

Thus, we can say that Speed of a particle moving with speed of light is same in all inertial reference frames.

Q.16 Initial positions of two particles are (4i+4j+7k)m and (2i+2j+5k)m respectively. If the velocity of first particle is 0.4 (i+j+k) m/sec, find the velocity of second particle so that they can collide after 10 sec.

Ans

Let the velocity of second particle is given by $\vec{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$

Hence the relative velocity between two particles is

$\vec{v} = (v_x - 0.4)\boldsymbol{i} + (v_y - 0.4)\boldsymbol{j} + (v_z - 0.4)\boldsymbol{k}$

The distance between two particles is = (2i+2j+5k) - (4i+4j+7k) = -2i-2j-2k

Now the particles can collide after 10 sec, hence

$$-2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = 10\,(v_x - 0.4)\boldsymbol{i} + (v_y - 0.4)\boldsymbol{j} + (v_z - 0.4)\boldsymbol{k}$$

On comparing the coefficient of I,j and k, we get

$v_x = 0.2, v_y = 0.2, v_z = 0.2$

Thus the velocity of the second particle is = 0.2 (i+j+k) m/sec

Multiple Choice Questions

1. Einstein's theories of relativity
a. are a result of the contradiction between Newton's laws of motion and Maxwell's equations.
b. proved the laws of nature are not universally applicable.
c. indicated that Maxwell's equations were in error.
d. showed that the speed of light is not always a constant.
e. none of the above
Answer: a

2. Special relativity is applied to
a. reference frames accelerating.
b. reference frames that do not accelerate.
c. very difficult mathematical problems.
d. general cases of relativity.
e. cases in which the object moves faster than the speed of light.
Answer: b

3. According to the theory of relativity,
a. as the speed of light is approached, time slows down and approaches zero.
b. as the speed of light is approached, time speeds up and approaches infinity.
c. as the speed of light is approached, distance shrinks and approaches zero.
d. a & c
e. none of the above
Answer: d

4. Which of the following scientific instruments has been used to document time dilation?
a. quartz crystals
b. laser
c. Hubble telescope
d. NIMBUS satellite
e. stopwatch

Answer: b

5. What is the difference between general and special relativity?
a. General relativity applies to accelerating frames of reference and special relativity does not.
b. Special relativity is a special case for two frames of reference accelerating in opposite directions.
c. General relativity can only solve problems of time dilation and length contraction; special relativity can apply to problems of large mass and the speed of light.
d. Special relativity is much more complex than general relativity, requiring rigorous mathematical calculations to explain.
e. Two of the above choices are true.
Answer: a

6. The Lorentz factor is a number a. more than one.
b. showing the magnitude of change in time and scale.
c. comparing the relative speed of two atomic clocks.
d. representing the atomic mass of a type of oil.
e that is used in Newtonian motion calculations.
Answer: b

10. Why is the speed of light special in Einstein's theories?
a. The speed of light is a constant built into Maxwell's equations.
b. The speed of light is relative to the observer's frame of reference.
c. All scientists agree that the speed of light can never be exceeded.
d. In relativity the moving clocks slowdown in time and light becomes dimmer.
e. none of the above
Answer: a

11. The theory of relativity resolves the paradox between
a. Newton's laws of motion and gravity.
b. time dilation and electromagnetism.
c. Maxwell's equations and time dilation.
d. Newton's laws of motion and Maxwell's equations.
e. gravity and time dilation.
Answer: d

12. In which direction does the length contraction phenomenon apply?
a. toward the center of the object
b. along the axis of motion
c. perpendicular to the axis of motion
d. in all directions simultaneously

e. in concentric circles
Answer:b

13. How is the mass of an object related to its energy?
a. The object's velocity is equal to the speed of light times the mass divided by the Lorentz number.
b. The object's rest mass is equal to the gravitational constant times the object's rest mass times the speed of light squared.
c. The object's rest energy is equal to the object's rest mass multiplied by the gravitational constant.
d. The object's rest mass is equal to the object's rest energy multiplied by the speed of light squared.
e. The object's rest energy is equal to the object's rest mass multiplied by the speed of light squared.
Answer: e

14. Gravitational red shift, one prediction of general relativity,
a. has been confirmed by experiments using laser light.
b. states that light coming from the Earth's surface to space will appear slightly redder in space than they do on Earth.
c. states that light coming to the Earth's surface from space will appear slightly bluer on Earth than they do in space.
d. can be explained using the Doppler effect.
e. all of the above
Answer: e

15. Pick out the false statement, using relativity principles.
a. The fact that mass and energy are equivalent can be verified in a nuclear power plant.
b. Relativity states that nothing can travel faster than the speed of light.
c. An object accelerated to the spee
d of light will appear to contract.
d. Einstein's relativity principles encompass Newton's laws of motion rather than replace them.
e. All the above are true.
Answer: b

16. In which of the following instances have the predictions of general relativity been confirmed?
a. gravitational bending of light
b. planetary orbits
c. gravitational red shift

d. a & c
e. a, b & c
Answer: e

17. Which of these follows from the theory of relativity?
a. Mass is relative to one's frame of reference.
b. Time is relative to one's frame of reference.
c. Distance is relative to one's frame of reference.
d. all of the above
e. none of the above
Answer: d

Unit 2

Conservation Laws

Q.1 Prove that the total kinetic energy of a system of many particles is equal to the sum of kinetic energy of center of mass and total kinetic energy of particles about the center of mass.

Ans Let us consider a system of n particles of mass m_1, m_2,……….. m_n, and position vectors r_1, r_2,……., r_n, and the total mass M which remains constant throughout the motion. Here

$$M = m_1 + m_2 + \dots \dots \dots \dots + m_n$$

And

$$M\overrightarrow{r_{cm}} = m_1 \overrightarrow{r_1} + m_2 \overrightarrow{r_2} + \dots \dots \dots \dots + m_n \overrightarrow{r_n} = \sum_{i=1}^{n} m_i \overrightarrow{r_i}$$

On differentiating this equation with respect to t, we get

$$M\overrightarrow{v_{cm}} = m_1 \overrightarrow{v_1} + m_2 \overrightarrow{v_2} + \dots \dots \dots \dots + m_n \overrightarrow{v_n} = \sum_{i=1}^{n} m_i \overrightarrow{v_i} = \vec{P}$$

And On again differentiating above equation with respect to t, we get

$$M\overrightarrow{a_{cm}} = m_1 \overrightarrow{a_1} + m_2 \overrightarrow{a_2} + \dots \dots \dots \dots + m_n \overrightarrow{a_n} = \sum_{i=1}^{n} m_i \overrightarrow{a_i} = \frac{d\vec{P}}{dt}$$

Now from Newton's second law, the external force acting on the system is given by

$$\vec{F} = M\overrightarrow{a_{cm}} = \sum_{i=1}^{n} m_i \overrightarrow{a_i} = M\overrightarrow{a_{cm}}$$

From these equations it is clear that the total external force on a system of particles i.e. vector sum of the external force acting on the system may be supposed to be acting on the centre of mass where whole mass is concentrated.

Since for each internal forces acting on some other particle, there is an equal and opposite force on anyone particle. Hence for a system as a whole the net internal force is zero. Thus the internal forces which the particles exert on each other has no role in motion of the centre of mass. i.e.

$\vec{F} = 0, and\ \overrightarrow{a_{cm}} = 0\ which\ implies\ that\ \overrightarrow{v_{cm}} = constant$

Now the above equation for momentum can be rewritten as

$$M\overrightarrow{v_{cm}} = \sum_{i=1}^{n} m_i \vec{v_i} = \sum_{i=1}^{n} m_i (\overrightarrow{v'_i} + \overrightarrow{v_{cm}})$$

Where $\overrightarrow{v'_i}$ is the velocity of ith particle in centre of mass system such that

$$\sum_{i=1}^{n} m_i \overrightarrow{v'_i} = \sum_{i=1}^{n} \overrightarrow{P'_i} = 0$$

i.e. the centre of mass of the frame, the total momentum of any system is zero.

Now total kinetic energy of a system of many particles is given by:

$$KE = \frac{1}{2}\sum_{i=1}^{n} m_i \overrightarrow{v_i}^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\overrightarrow{v'_i} + \overrightarrow{v_{cm}}\right)^2$$

Or

$$KE = = \frac{1}{2}\sum_{i=1}^{n} m_i \left({v'_i}^2 + {v'_{cm}}^2 + 2\overrightarrow{v'_i}\overrightarrow{v_{cm}}\right)$$

$$KE = = \frac{1}{2}\sum_{i=1}^{n} m_i\ {v'_i}^2 + \sum_{i=1}^{n} m_i\ {v'_{cm}}^2 + 2\sum_{i=1}^{n} m_i\ \overrightarrow{v'_i}\overrightarrow{v_{cm}}$$

But since

$$\sum_{i=1}^{n} m_i \overrightarrow{v'_i} = \sum_{i=1}^{n} \overrightarrow{P'_i} = 0$$

Hence

$$KE = = \frac{1}{2}\sum_{i=1}^{n} m_i\ {v'_i}^2 + \sum_{i=1}^{n} m_i\ {v'_{cm}}^2$$

i.e. the total kinetic energy of a system of many particles is equal to the sum of kinetic energy of center of mass and total kinetic energy of particles about the center of mass.

Q.2 If the electrical potential $V = 6x - 8x^2y + 7y^2 + 13\,yz - 4z$ volts, then calculate the gradient of electric potential.

Ans

$$\text{grad V} = \nabla V = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)(6x - 8x^2y + 7y^2 + 13\,yz - 4z)$$

$$\nabla V = \frac{\partial}{\partial x}(6x - 8x^2y + 7y^2 + 13\,yz - 4z)\boldsymbol{i} + \frac{\partial}{\partial y}(6x - 8x^2y + 7y^2 + 13\,yz - 4z)\boldsymbol{j} + \frac{\partial}{\partial z}(6x - 8x^2y + 7y^2 + 13\,yz - 4z)\boldsymbol{k}$$

$$\nabla V = (6 - 16x)\boldsymbol{i} + (-8x^2 + 14y + 13z)\boldsymbol{j} + (13y - 4)\boldsymbol{k}$$

Q.3 Prove that for conservative force:- F= - grad U = -$\vec{\nabla}$U, and curl $\vec{F} = \vec{\nabla} \times \vec{F} = 0$

Ans A force acting on a particle is said to be conservative if the work done by a force in moving a particle from one point to another is independent of the path.

If **F** is conservative force, then according to definition of potential energy

$$\int_P^Q \boldsymbol{F}.\boldsymbol{dr} = V_P - V_Q$$

If dV is the increase in potential energy of a particle when it suffers a displacement d**r** under the influence of conservative force, then we have

$$\int_P^Q \boldsymbol{F}.d\boldsymbol{r} = -\int_P^Q dV$$

Or $dV = -\boldsymbol{F}.d\boldsymbol{r}$

This equation simply indicates that the work done by conservative force is equal to the corresponding decrease in potential energy of the particle.

The displacement d**r** may be expressed as

$$d\boldsymbol{r} = \boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz$$

Now total increase in potential energy of the particle as a result of displacement d**r** can be expressed as

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

This equation can be written as

$$dV = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)V.(\boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz)$$

$$dV = (\nabla V).d\boldsymbol{r}$$

Where $\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$

Now,

$$\boldsymbol{F}.d\boldsymbol{r} = -(\nabla V).d\boldsymbol{r}$$

Or

$$\boldsymbol{F}.d\boldsymbol{r} + (\nabla V).d\boldsymbol{r} = 0$$

$$(\boldsymbol{F} + \nabla V).d\boldsymbol{r} = 0$$

Since this equation is true for all arbitrary infinitesimal displacement d**r**, Therefore we get

$$(\boldsymbol{F} + \nabla V) = 0$$

Or

$$\boldsymbol{F} = -\nabla V = -grad\ V$$

Now

Curl **F** = curl (- grad V) = - curl grad V = $-\nabla X(\nabla V)$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) X \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) V$$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) X \left(\frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k}\right)$$

$$= -\begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial z} & \frac{\partial V}{\partial z} \end{matrix}$$

$$= -\left[\boldsymbol{i}\left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right) - \boldsymbol{j}\left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x}\right) - \boldsymbol{k}\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right)\right]$$

As V is perfect differential, so

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}, \qquad \frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}, \qquad \frac{\partial^2 V}{\partial z \partial x} = \frac{\partial^2 V}{\partial x \partial z}$$

Hence Curl **F** = 0

i.e. curl of a conservative force is always zero.

Q.4 Prove that the force F= $\{(2xy + z^2)\boldsymbol{i} + x^2\boldsymbol{j} + 2xz\boldsymbol{k}\}$ is conservative. Find:-

(i) Potential energy function for above force

(ii) The work done to displace a particle from point (0,1,2) to point (5,2,7)

Ans

$$curl\,\boldsymbol{F} = \nabla X\,\boldsymbol{F} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) X\{(2xy + z^2)\boldsymbol{i} + x^2\boldsymbol{j} + 2xzk\}$$

$$\nabla X\,\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + z^2) & x^2 & 2xz \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}2xz - \frac{\partial}{\partial z}x^2\right\}\boldsymbol{i} - \left\{\frac{\partial}{\partial x}2xz - \frac{\partial}{\partial z}(2xy + z^2)\right\}\boldsymbol{j}$$

$$+ \left\{\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(2xy + z^2)\right\}\boldsymbol{k}$$

$$\nabla X\,\boldsymbol{F} = (0-0)\boldsymbol{i} - (2z - 2z)\boldsymbol{j} + (2x - 2x)\boldsymbol{k} = 0$$

$$curl\,\boldsymbol{F} = \nabla X\,\boldsymbol{F} = 0$$

Which is the necessary and sufficient condition for a force to be conservative.

Hence the force is conservative.

Now for a conservative force, we have

$$\boldsymbol{F} = -grad\,\boldsymbol{V} = -\nabla V$$

i.e.

$$(2xy + z^2)\boldsymbol{i} + x^2\boldsymbol{j} + 2xz\boldsymbol{k} = -\left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)V$$

On comparing the coefficient of I, j and k, we have

$$\frac{\partial V}{\partial x} = -(2xy + z^2)$$

$$\frac{\partial V}{\partial y} = -x^2$$

And

$$\frac{\partial V}{\partial z} = -2xz$$

On solving these equations, we get

$$V = -x^2y - z^2x$$

Now work done in moving the particle from a point to another is given by:

$$W = \int_{r_1}^{r_2} \vec{F}.\overrightarrow{dr} = \int_{r_1}^{r_2} \{(2xy + z^2)\boldsymbol{i} + x^2\boldsymbol{j} + 2xz\boldsymbol{k}\}.(\boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz)$$

$$= \int_{r_1}^{r_2} (2xy + z^2)dx + x^2dy + 2xzdz$$

$$W = [x^2y + z^2x + x^2y + z^2x]_{r_1}^{r_2}$$

For r_1, $x_1 = 0, y_1 = 1$ and $z_1 = 2$, and for r_2, $x_2 = 5, y_2 = 2$ and $z_2 = 7$

On putting these values, we get

$$W = 295\ unit$$

Q.6 Prove that the angular momentum of a system of particles with respect to O is the vector sum of the angular momentum of the center of mass with respect to O and angular momentum of the system of particles with respect to center of mass.

$$\overrightarrow{\boldsymbol{J_o}} = \overrightarrow{\boldsymbol{J_{cmo}}} + \overrightarrow{\boldsymbol{J_{cm}}}$$

Ans The angular momentum of i^th^ particle about the centre of mass is

$= (\boldsymbol{r_i} - \boldsymbol{r_{cm}})\, X\, m_i(\boldsymbol{v_i} - \boldsymbol{v_{cm}})$

Where $\boldsymbol{r_i}$ and $\boldsymbol{v_i}$ are the position vector and velocity of ith particle relative to any arbitrary origin O respectively where $\boldsymbol{r_{cm}}$ and $\boldsymbol{v_{cm}}$ are the position vector and velocity of center of mass relative to origin O.

The total angular momentum of a system of particles about their centre of mass can be expressed as

$$J_{cm} = \Sigma(\boldsymbol{r_i} - \boldsymbol{r_{cm}})\, X\, m_i(\boldsymbol{v_i} - \boldsymbol{v_{cm}})$$

$$J_{cm} = \sum_i \boldsymbol{r_i}\, X\, m_i\boldsymbol{v_i} - \sum_i \boldsymbol{r_i}\, X\, m_i\boldsymbol{v_{cm}} - \sum_i \boldsymbol{r_{cm}}\, X\, m_i\boldsymbol{v_i} + \sum_i \boldsymbol{r_{cm}}\, X\, m_i\boldsymbol{v_{cm}}$$

$$J_{cm} = \sum_i \boldsymbol{r_i}\, X\, \boldsymbol{p_i} - \sum_i m_i\boldsymbol{r_i}\, X\, \boldsymbol{v_{cm}} - \boldsymbol{r_{cm}}\, X \sum_i m_i\boldsymbol{v_i} + \boldsymbol{r_{cm}}\, X\, \boldsymbol{v_{cm}} \sum_i m_i$$

But from the definition,

$$\sum_i m_i\boldsymbol{r_i} = \boldsymbol{r_{cm}} \sum_i m_i = M\boldsymbol{r_{cm}}$$

$$\sum_i m_i\boldsymbol{v_i} = \boldsymbol{v_{cm}} \sum_i m_i = M\boldsymbol{v_{cm}}$$

Where $\sum_i m_i = M$ is the total mass of the system.

Also

$\sum_i \boldsymbol{r_i}\, X\, \boldsymbol{p_i} = \boldsymbol{J}$ is the total angular momentum of the system about origin O.

Using these substitutions we can write the above equation as

$$\boldsymbol{J}_{cm} = \boldsymbol{J} - M\boldsymbol{r_{cm}} X\, \boldsymbol{v_{cm}} - \boldsymbol{r_{cm}}\, X\, M\, \boldsymbol{v_{cm}} + \boldsymbol{r_{cm}}\, X\, M\, \boldsymbol{v_{cm}}$$

$$\boldsymbol{J}_{cm} = \boldsymbol{J} - M\boldsymbol{r_{cm}} X\, \boldsymbol{v_{cm}}$$

$$\boldsymbol{J} = \boldsymbol{J}_{cm} + \boldsymbol{r_{cm}} X\, (M\boldsymbol{v_{cm}})$$

$$\boldsymbol{J} = \boldsymbol{J}_{cm} + \boldsymbol{r_{cm}} X\, \boldsymbol{P}$$

$$\boldsymbol{J} = \boldsymbol{J}_{cm} + \boldsymbol{J}_{cmo}$$

Here the term $\boldsymbol{r_{cm}} X\, \boldsymbol{P} = \boldsymbol{J}_{cmo}$ represent the angular momentum\about the same arbitrary point O of the whole mass of the system concentrate at the centre of mass.

The above equation gives a relation between the total angular momentum of a system of particles about any fixed point O and the internal angular momentum $\boldsymbol{J}_{cm}$.

Q.7 The weight of empty rocket is 5000 kg. In this rocket 40,000 kg fuel is filled up. If exhaust velocity of fuel is 2 km/sec, then find out the maximum velocity of rocket.

$[\boldsymbol{log_e\, 10 = 2.3, log_{10}\, 3 = 0.4771}]$

Ans According to the given problem

M_0 = 5000 kg, M= 5000 kg+40,000 kg = 45000kg

V_{exh}= 2 km/sec

Maximum velocity of the rocket is given by

$$V = -V_{exh} log_e \frac{M_0}{M}$$

$$V = -2\, log_e \frac{5000}{45000} = -2\, log_e \frac{1}{9} = 4\, log_e 3 = 4\, \frac{log_{10} 3}{log_{10} e} = 4 log_{10} 3\, X\, log_e 10$$

$$V = 4\, X\, 0.4771 X\, 2.3 = 4.3893\, Km/sec$$

Q.8 **Define gravitational field, intensity of gravitational field, gravitational potential energy, and gravitational potential. Prove that the gravitational potential energy of two particles of mass m1 and m2 placed at a distance r is $\boldsymbol{\frac{Gm_1m_2}{r}}$**

Ans **Gravitational field:** Every particle of matter exerts a force of attraction on every other particle. This force is called gravitational force of attraction and is given by Newton's law of gravitation. The area round about the attracting particle within which its gravitational force of attraction is perceptible is called its gravitational field.

Intensity of gravitational field: The intensity of gravitational field or the gravitational attraction at a point in a gravitational field is the force experienced by unit mass placed at that point in the field. Thus the gravitational field at a point due to a particle of mass M at a distance r from the point is, given by

$\boldsymbol{f_g = -\frac{GM}{r^2}\,\hat{r}}$

Where $\boldsymbol{\hat{r}} = \frac{\boldsymbol{r}}{r}$ is a unit vector along **r**.

Thus force per unit mass is a measure of the field intensity.

Gravitational potential energy:

The gravitational potential energy of a system of particles is generally defined as the amount of work that must be done by an external agent to assemble the system starting from an infinite distance to the present position against the gravitational field. The potential energy of the particles is assumed to be zero when they are infinitely apart.

Consider a system of two masses m_1 and m_2. Let them initially be infinitely far from each other. Let us assume that mass m_1 is already present and we bring m_2 from infinity to at the point where mass m_1 is situated. Then the work required to be done by the gravitational force exerted by m_1 on m_2 is

$$W_{12} = -\int_{\infty}^{r_{12}} \boldsymbol{F_{21}}.d\boldsymbol{r} = -\int_{\infty}^{r_{12}} \left(\frac{Gm_1m_2}{r^2}\,\hat{\boldsymbol{r}}\right).dr = \int_{\infty}^{r_{12}} \frac{Gm_1m_2}{r^2}.dr = \left[-\frac{Gm_1m_2}{r}\right]_{\infty}^{r_{12}} = -\frac{Gm_1m_2}{r_{12}}$$

The gravitational potential energy of two particles of mass m1 and m2 placed at a distance r is $\frac{Gm_1m_2}{r}$

If we want to separate the system into isolated masses again, we would have to supply the same amount of energy. Hence the gravitational potential energy of two particles of mass m1 and m2 placed at a distance r is given by $\frac{Gm_1m_2}{r}$

Gravitational potential: If a body is moved in the gravitational field of another body, a certain amount of work is to be done. If moved in the direction of the field, the work is done by the field itself, and if moved against the field, the work is to be done by some external agent.

"The work done in moving a unit mass from infinity to any point in the gravitational field of a body is called the gravitational potential at that point due to the body."

Gravitational potential at a point in gravitational field may also be defined as the potential energy of unit mass placed at that point. It is denoted by V_g. Its value at a distance r from the body of mass M is given by

$$V_g = -\int_{\infty}^{r} \boldsymbol{f_g}.d\boldsymbol{r} = -\int_{\infty}^{r} \left(\frac{GM}{r^2}\,\hat{\boldsymbol{r}}\right).dr = \int_{\infty}^{r} \frac{GM}{r^2}.dr = \left[-\frac{GM}{r}\right]_{\infty}^{r} = -\frac{GM}{r}$$

It must be noted that potential is a scalar quantity while gravitational field is a vector quantity.

Q.9 Describe the principle of rocket. Derive the following relation for a rocket, where symbols have their usual meanings.

$$\boldsymbol{V = V_0 + V_r \log_e \frac{M_0}{M} - gt}$$

Ans A rocket in its simplest form is a chamber enclosing a gas (fuel and oxidizing agent) under pressure. A small opening at one end of the chamber allows the gas to escape, and in doing so provides a thrust that propels the rocket in the opposite direction. The oxidizing agent of oxygen is carried in the liquid form of oxidizers like H_2O_2 and HNO_3. When the fuel burns, a jet of hot gases emerges forcefully from the tail of the rocket which may be considered as action force. A force is exerted by the jet of hot gases on the rocket which is reaction force. The jet of hot gases acquires momentum in the backward direction while rocket due to reaction force acquires momentum in the forward direction, consequently the rocket gains forward motion. The total momentum of the system is conserved as there is no external force.

Let M_0 is the initial mass of the rocket-fuel system and M is the mass after time t when its velocity is V relative to earth. Let m be the rate at which mass of the gases leave the rocket with exhaust velocity of gases V_r (downward) relative to rocket.

Since the mass of rocket-fuel system decreases due to burning if fuel, the rate of change of mass of rocket

$= \frac{dm}{dt} = -m$ (Since decrease in mass of the rocket = increase in mass of the ejected gas)

The moment of rocket after time t = MV

So, the rate of change of momentum of rocket + unburnt fuel = $\frac{d}{dt}(MV)$

Hence $\left(\frac{\Delta p}{dt}\right)_{rocket} = M\frac{dV}{dt} + V\frac{dM}{dt}$

Velocity of gases relative to earth = $V-V_r$

Rate of change of momentum of gases

$$\left(\frac{\Delta p}{dt}\right)_{rocket} = m\,(V - V_r) = -\frac{dM}{dt}\,(V - V_r)$$

Thus, Rate of change of momentum of system is obtained by

$$\left(\frac{\Delta p}{dt}\right)_{System} = M\frac{dV}{dt} + V\frac{dM}{dt} - \frac{dM}{dt}(V - V_r)$$

$$\left(\frac{\Delta p}{dt}\right)_{System} = M\frac{dV}{dt} + V\frac{dM}{dt} - V\frac{dM}{dt} + V_r\frac{dM}{dt}$$

$$\left(\frac{\Delta p}{dt}\right)_{System} = M\frac{dV}{dt} + V_r\frac{dM}{dt}$$

The resultant rate of change of momentum is equal to the resultant external force F acting on the system i.e.

$$F = M\frac{dV}{dt} + V_r\frac{dM}{dt}$$

Here $u\frac{dM}{dt}$ us the thrust on the rocket which is opposite to the velocity u.

When gravitational force is taken into account, then F = -Mg

i.e. $M\frac{dV}{dt} + V_r\frac{dM}{dt} = -Mg$

or $\frac{dV}{dt} + \frac{V_r}{M}\frac{dM}{dt} = -g$

$$dV + V_r\frac{dM}{M} = -gdt$$

On integrating this equation, we get

$$V + V_r\, log_e M = -gt + C$$

Where C is an arbitrary constant and can be determined by initial condition

i.e. AT V= 0 , M= M_0

Which gives $C = V_r\, log_e M_0$

Hence

$V + V_r\, log_e M = -gt + u\, log_e M_0$

Or $V = -V_r\, log_e M - gt + u\, log_e M_0$

Or $V = V_r\, log_e \frac{M_0}{M} - gt$

Thus the effect of gravitational force is to decrease the speed of the rocket. If at any instant t, the rocket has some initial velocity V_0, then after time t, its velocity is given by

$$V = V_0 + V_r\, log_e \frac{M_0}{M} - gt$$

Q.10 Show that the force acting on the system is conservative if the mechanical energy of the system is conserved.

Ans Let a conservative force $\vec{F}$ acting on a particle displaces it from position $\vec{r_1}$ to position $\vec{r_2}$. The work done according to work energy theorem will be equal to the increase in the kinetic energy i.e.

$W = \int_{r_1}^{r_2} \vec{F}.\overrightarrow{dr} = \frac{1}{2}\, m\, v_2^2 - \frac{1}{2}\, m\, v_1^2$

Where v_1 and v_2 are the velocities of the particle at position$\vec{r_1}$ and $\vec{r_2}$.

Since the force is acting on this given region of space, its linear integral in moving from $\vec{r_1}$ to $\vec{r_2}$ must be independent of the path followed by the particle. This line integral must be equal to the change in the potential energy of the particle i.e.

$\int_{r_1}^{r_2} \vec{F}.\overrightarrow{dr} = U_1 - U_2$

Thus we have,

$$\frac{1}{2}\, m\, v_2^2 - \frac{1}{2}\, m\, v_1^2 = U_1 - U_2$$

Or

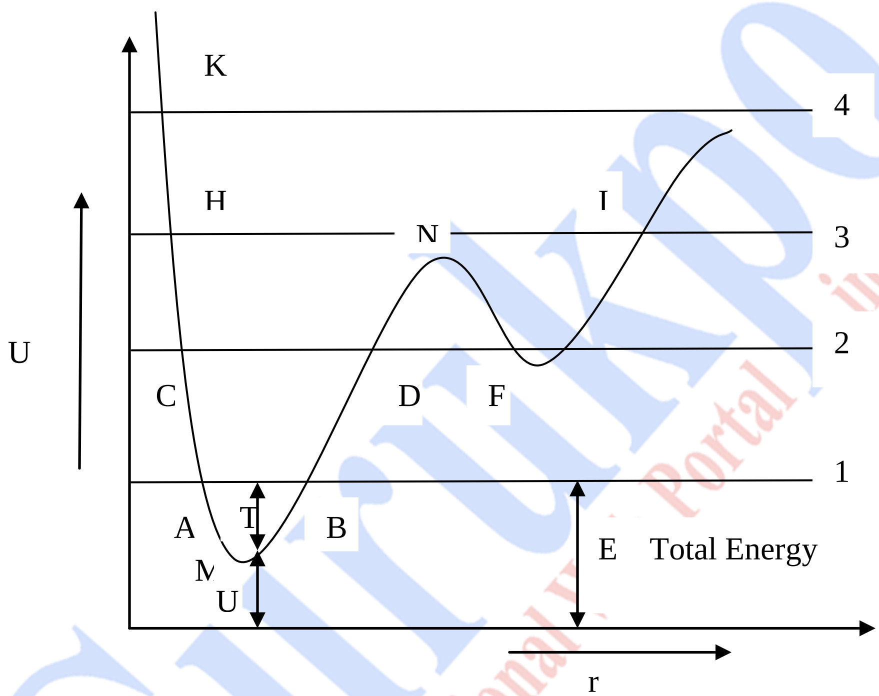
$$U_1 + \frac{1}{2}\, m\, v_1^2 = \frac{1}{2}\, m\, v_2^2 + U_2$$

i.e. the sum of the kinetic and potential energies of the particle at eny point under the influence of this force is constant. This quantity is known as energy function E.

$$E = \frac{1}{2}\, m\, v^2 + U = constant$$

This is known as the law of conservation of energy which states that for the conservative forces the sum of kinetic and potential energies of the particle i.e. the total mechanical energy is always conserved.

Q.11 Discuss the potential energy curve for the motion of a particle. Explain the position of stable and unstable equilibrium. What is region of neutral equilibrium?



Ans The potential energy U of a particle in a force field is a function of space. A curve showing the variation of potential function with the position of the point under consideration is known as the potential energy curve. The figure above shows the variation of potential energy of a particle versus its distance from the origin along the line of its motion.

The force on the particle is given by

$$F = -\frac{dU}{dr}$$

Which means U decreases with increase in r i.e. slope of the potential energy curve is negative; therefore force is positive i.e. force is towards positive r and vice versa. This means that a particle at any point experiences a force which tries to bring it to region of lower potential energy.

Now from the conservation of energy we have,

$$E = \frac{1}{2}\, m\, v^2 + U$$

Or

$$v = \pm\sqrt{\frac{2}{m}\,(E - U)}$$

Thus, in order to v to be real, we must have E > U i.e. the motion to take place, the total energy must be greater than the potential energy.

The particles of total energy E is shown by line 1 can only exist in between A and B on the potential energy curve. At these points the kinetic energy and hence the velocity of the particle is zero and it turns back under the influence of the force tending to bring it towards the minimum potential energy point M. Points A nad B are called turning points. The particle oscillates between point A and B i.e. particle is bound and can not escape unless it gets energy from outside.

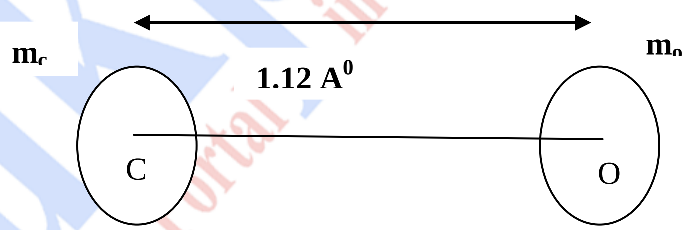
If the particle has total energy E, it can exist in two regions: either between point C and D or between F and G. But once it is trapped in one region, there is no possibility to transfer to othe region. The particle thus has a possibility to having oscillation in either of the bounded regions CD or FG.

There are some special points on the curve where slope of the potential energy curve is zero i.e. particle at these points experiences no force and particle is in state of equilibrium. However the points where potential energy is maximum is called unstable equilibrium position because any slight displacement of the particle on either side causes a force to act on the particle which tries to further displacement away from the point.

On the other hand when a particle occupies a position where potential energy is minimum, is called stable equilibrium position because any slight displacement of the particle on either side causes a force to act on the particle which tries to restore it back towards that point.

The state of equilibrium of a particle when in a region all over which the potential energy is constant is called neutral equilibrium.

Q.12 In a carbon mono oxide molecule, the distance between the carbon (m_c= 12 units) and oxygen (m_o = 16 units) is 1.12 A^0. Find the position of center of mass of molecule with respect to carbon atom.



Ans As we know that the centre of mass of two particles must lie on the line joining them and the ratio of distances of the two particles from the centre of mass is the inverse ratio of their masses. i.e.

If x is the distance of the position of centre of mass from the C atom, then (1.12 –x) will be the distance of centre of mass from the oxygen.

Hence we have

$$\frac{x}{1.12 - x} = \frac{16}{12}$$

Or

$$12x = 16(1.12 - x) = 17.92 - 16\,x$$

$$28x = 17.92$$

$$x = 0.64\ \text{A}^0$$

Q.13 A particle of mass 20 g is moving in a circle of 4 cm radius with constant speed of 10 cm/sec. What is its angular momentum about (i) the center of the circle (ii) a point of the axis of the circle 3 cm distant from the centre?

Ans

(i) The angular momentum of a particle of mass m is given by

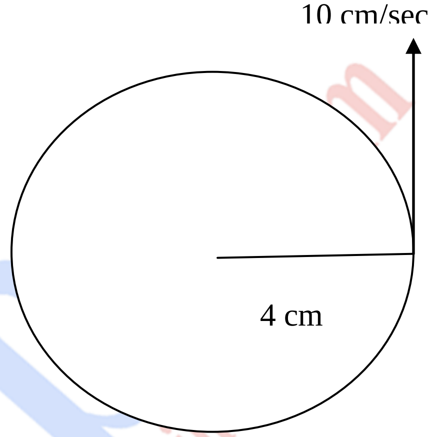
$$\vec{L} = m\,(r\;X\;v)$$

Now for the angular momentum about the centre of the circle, r = 4 cm and v = 10cm/sec and r and v are perpendicular to each other.

So

$\vec{L} = 20\,(4X\;10) =$

$800\;g\;cm^2\;sec\quad along\;the\;axis\;perpendicul$
$plane\;\;of\;the\;circle\;and\;passes\;through\;centre$

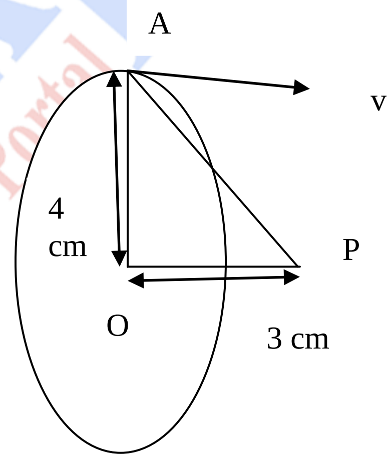


Now for the angular momentum about a point of the axis of the circle 3 cm distant from the centre, r = AP = 5 cm and v = 10cm/sec and r and v are perpendicular to each other.

So

$\vec{L} = 20\,(5X\;10) =$

$1000\;g\;cm^2\;sec\quad along\;the\;axis\;perpendicular\;to\;th$
$plane\;\;of\;r\;and\;AP$



Q.14 Discuss the advantages of multistage rocket over a single stage rocket.

Ans A **multistage** (or **multi-stage**) **rocket** is a rocket that uses two or more *stages*, each of which contains its own engines and propellant. A *tandem* or *serial* stage is mounted on top of another stage; a *parallel* stage is attached alongside another stage. The result is effectively two or more rockets stacked on top of or attached next to each other. Taken together these are sometimes called a launch vehicle. Two stage rockets are quite common, but rockets with as many as five separate stages have been successfully launched.

Advantages: The main reason for multi-stage rockets and boosters is that once the fuel is exhausted, the space and structure which contained it and the motors themselves are useless and only add weight to the vehicle which slows down its future acceleration. By dropping the stages which are no longer useful, the rocket lightens itself. The thrust of future stages is able to provide more acceleration than if the earlier stage were still attached, or a single, large rocket would be capable of. When a stage drops off, the rest of the rocket is still traveling near the speed that the whole assembly reached at burn-out time. This means that it needs less total fuel to reach a given velocity and/or altitude.

A further advantage is that each stage can use a different type of rocket motor each tuned for its particular operating conditions. Thus the lower stage motors are designed for use at atmospheric pressure, while the upper stages can use motors suited to near vacuum conditions. Lower stages tend to require more structure than upper as they need to bear their own weight plus that of the stages above them, optimizing the structure of each stage decreases the weight of the total vehicle and provides further advantage.

Disadvantages: On the downside, staging requires the vehicle to lift motors which are not being used until later, as well as making the entire rocket more complex and harder to build. In addition, each staging event is a significant point of failure during a launch, with the possibility of separation failure, ignition failure, and stage collision. Nevertheless the savings are so great that every rocket ever used to deliver a payload into orbit has had staging of some sort.

In more recent times the usefulness of the technique has come into question due to developments in technology. In the case of the Space Shuttle the costs of space launches appear to be mostly composed of the operational costs of the people involved, as opposed to fuel or equipment. Reducing these costs appears to be the best way to lower the overall launch costs. New technology that is mainly in the theoretical and developmental stages is being looked at to lower the costs of launch vehicles. More information can be found on single stage to orbit designs that do not have separate stages.

Separation of each portion of a multistage rocket introduces additional risk into the success of the launch mission. Reducing the number of separation events results in a reduction in complexity. Separation events occur when stages or strap-on boosters separate after use, when the payload fairing separates prior to orbital insertion, or when the launch escape system--used in many early human spaceflight missions--separates after the early phase of the launch.

Q.14 If center of mass of a system of three particles of mass 2g, 3g and 6g be at a point (1,1,1), then where should the fourth particle of mass 8g be placed so that the new center of mass be at (3,3,3)

Let the co-ordinate of the three particles are 2g (x_1,y_1, z_1), 3g(x_2,y_2, z_2) and 6g(x_3,y_3, z_3). The co-ordinate of centre of mass is (1,1,1), then

$1 = \frac{2x_1 + 3x_2 + 6\,x_3}{2+3+6}$, $1 = \frac{2y_1 + 3y_2 + 6\,y_3}{2+3+6}$ and $1 = \frac{2z_1 + 3z_2 + 6\,z_3}{2+3+6}$

or

$2x_1 + 3x_2 + 6\,x_3 = 11, 2y_1 + 3y_2 + 6\,y_3 = 11, 2z_1 + 3z_2 + 6\,z_3 = 11$

Now let fourth particle of mass 8g and co-ordinate (x_4,y_4, z_4), be placed so that the new center of mass be at (3,3,3)

Hence the equation of centre of mass becomes

$3 = \frac{2x_1 + 3x_2 + 6\,x_3 + 8\,x_4}{2+3+6+8}$, $3 = \frac{2y_1 + 3y_2 + 6\,y_3 + 8\,y_4}{2+3+6+8}$ and $3 = \frac{2z_1 + 3z_2 + 6\,z_3 + 8z_4}{2+3+6+8}$

Or

$2x_1 + 3x_2 + 6\,x_3 + 8\,x_4 = 57, 2y_1 + 3y_2 + 6\,y_3 + 8\,y_4 = 57, 2z_1 + 3z_2 + 6\,z_3 + 8z_4 = 57$

From the above two set of equations, we get

$11 + 8\,x_4 = 57, 11 + 8\,y_4 = 57, 11 + 8z_4 = 57$

Which gives

$x_4 = 5.75$, $y_4 = 5.75$ and $z_4 = 5.75$

Multiple Choice Questions

1. A body is moving along a circular path with variable speed. It has
   (a) a radial acceleration
   (b) a tangential acceleration
   (c) zero acceleration
   (d) both tangential and radial acceleration

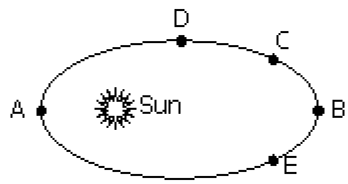
Answer: d

2. A body is traveling in a circle at constant speed. It
   (a) has constant velocity.
   (b) has no acceleration
   (c) has an inward acceleration
   (d) has an outward radial acceleration

Answer: c

3. A thief stole a box with valuable article of weight 'W' and jumped down a wall of height h. Before he reach the ground he experienced a load of
(a) zero
(b) W / 2
(c) W
(d) 2 W
Answer: a

3. The planet mercury is revolving in an elliptical orbit around the sun as shown in figure. The kinetic energy of mercury will be greater at



(a) A
(b) B
(c) C
(d) D
Answer: a

4. F_g and F_e represents gravitational and electrostatic forces respectively, between the two electrons situated at a distance of 10 m. The ratio F_g/F_e is of the order of
(a) 10^{43}
(b) 10^{36}
(c) 10^{-43}
(d) 10^{-36}
Answer: c

5. There is no atmosphere on the moon because
(a) it is closer ot the earth
(b) it revolves round the earth
(c) it gets light from the sun
(d) the escape velocity of gas molecules is less than their root mean square velocity here
Answer: d

6. If the earth is $1/4^{th}$ of its present distance from the sun, the duration of the year would be
(a) 1/4 of the present year
(b) 1/6 of the present year
(c) 1/8 of the present year
(d) 1/16 of the present year

Answer: c

5. The period of geostationary artificial satellite of earth is
(a) 6 hours
(b) 12 hours
(c) 24 hours
(d) 365 days
Answer: c

6. The period of a satellite in a circular orbit of radius R is T. The period of another satellite in circular orbit of radius 4R is
(a) T/4
(b) 8T
(c) 2T
(d) T/8
Answer: b

7. The distance of Neptune and Saturn from the sun are nearly 10^{13} m and 10^{12} m respectively. Assuming that they move in circular orbits, their periodic times would be in the ratio of
(a) 10
(b) 100
(c) 10 $\sqrt{10}$
(d) 1000
Answer: c

8. Kepler's second law regarding constancy of arial velocity of a planet is a consequence of the law of conservation of
(a) energy
(b) angular momentum
(c) linear momentum
(d) none of these
Answer: b

9. When the planet comes nearer the sun moves
(a) fast
(b) slow
(c) constant at every point
(d) none of the above
Answer: a

10. A satellite is revolving around the sun in a circular orbit with uniform velocity v. If the gravitational force suddenly disappears, the velocity of the satellite will be
(a) zero
(b) v

(c) 2v
(d) infinity
Answer: b

11. Who among the following first gave the experimental velocity of G?
(a) Cavendish
(b) Copernicus
(c) Brook Taylor
(d) none of these
Answer: a

12. The dimensions of universal gravitational constant are
(a) $M^2 L^2 T^{-2}$
(b) $M^{-1} L^3 T^{-2}$
(c) $M L^{-1} T^{-2}$
(d) $M L^2 T^{-2}$
Answer: b

13. Geo-stationary satellite
(a) revolves about the polar axis
(b) has a time period less than that of the earth's satellite
(c) moves faster than a near earth satellite
(d) is stationary in the space
Answer: a

Unit 3

Rigid Body Dynamics

Q.1 What are the inertial coefficients? Derive the expression for these inertial coefficients. Explain the term moment of inertia and product of inertia.

Ans

Angular momentum oa a system of particles is given by the equation

$$\boldsymbol{J} = \sum_i \boldsymbol{r_i}\, X\, \boldsymbol{p_i} = \sum_i \boldsymbol{r_i}\, X\, m_i\, \boldsymbol{v_i} = \sum_i m_i\, \boldsymbol{r_i}\, X\, \boldsymbol{v_i}$$

Where i stands for ith particle.

$\boldsymbol{r_i}$ **=** position vector of i^th^ partice

$\boldsymbol{p_i} = m_i\boldsymbol{v_i}$ **=** linear momentum of i^th^ particle

m_i = mass of i^th^ particle

$\boldsymbol{v_i}$ **=** velocity of i^th^ particle

If $\vec{\omega}$ is the angular velocity of the system, then

$\boldsymbol{v_i} = \vec{\omega}\, \boldsymbol{X}\, \boldsymbol{r_i}$

Hence, Angular momentum $\boldsymbol{J} = \sum_i m_i\, \{\boldsymbol{r_i}\, X\, (\vec{\omega}\, \boldsymbol{X}\, \boldsymbol{r_i})\}$

$$\boldsymbol{J} = \sum_i m_i\, [(\boldsymbol{r_i}.\boldsymbol{r_i})\, \omega - (\boldsymbol{r_i}.\omega)\boldsymbol{r_i}\,]X\, (\vec{\omega}\, \boldsymbol{X}\, \boldsymbol{r_i})\}$$

In the forms of component **J,** $\boldsymbol{r_i}$ and $\vec{\omega}$ is written as

$$\boldsymbol{J} = \boldsymbol{i}\, J_x + \boldsymbol{j}\, J_y + \boldsymbol{k}\, J_z$$

$$\boldsymbol{r_i} = \boldsymbol{i}\, x_i + \boldsymbol{j}\, y_i + \boldsymbol{k}\, z_i$$

$$\vec{\omega} = \boldsymbol{i}\, \omega_x + \boldsymbol{j}\, \omega_y + \boldsymbol{k}\, \omega_z$$

On substituting these values in above equation, we have

$$J_x = \omega_x \sum_i m_i(\, r_i{}^2 - \; x_i{}^2) - \omega_y \sum_i m_i x_i y_i - - \omega_z \sum_i m_i x_i z_i$$

$$J_y = -\omega_x \sum_i m_i x_i y_i + \omega_y \sum_i m_i (r_i^2 - y_i^2) - \omega_z \sum_i m_i y_i z_i$$

$$J_z = -\omega_x \sum_i m_i x_i z_i - \omega_y \sum_i m_i y_i z_i + \omega_z \sum_i m_i (r_i^2 - z_i^2)$$

From the equations, it is clear that the components of angular momentum are linear functions of components of angular velocity. In these equations, the coefficients of ω_x , ω_y and ω_z are called inertia coefficients. This equation representing linear transformation may be expressed in matrix form as follows:

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

In brief, the above matrix equation can be written in tensor form as

$$|\boldsymbol{J} >= I\, |\vec{\omega} >$$

Where I is the moment of inertia tensor. It has nine components

$I_{xx}, I_{xy}, I_{xz}, I_{yx}, I_{yy}, I_{yz}, I_{zx}, I_{zy}$ and I_{zz}

The components I_{xx}, I_{yy}, I_{zz} are given by

$$I_{xx} = \sum_i m_i (r_i^2 - x_i^2)$$

$$I_{yy} = \sum_i m_i (r_i^2 - y_i^2)$$

$$I_{zz} = \sum_i m_i (r_i^2 - z_i^2)$$

are called moment of inertia and other components

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i$$

$$I_{xz} = I_{zx} = -\sum_i m_i x_i z_i$$

$$I_{yz} = I_{zy} = -\sum_i m_i y_i z_i$$

are called product of inertia.

If we choose the axes of body such that all product of inertia become zero i.e.

$$I_{xy} = I_{xz} = I_{yx} = I_{yz} = I_{zx} = I_{zy} = 0$$

Then the axes of the body are called principle moment of inertia. They are denoted by I_1, I_2, I_3.

Q.2 Explain the principal axes with example.

Ans The use of Newton's second law for rotation involves the assumption that the axis about which the rotation is taking place is a principal axis. Since most common rotational problems involve the rotation of an object about a symmetry axis, the use of this equation is usually straightforward, because axes of symmetry are examples of principle axes. A principal axis may be simply defined as one about which no net torque is needed to maintain rotation at a constant angular velocity. The issue is raised here because there are some commonly occurring physical situations where the axis of rotation is not a principal axis. For example, if your automobile has a tire which is out of balance, the axle about which it is rotating is not a principal axis. Consequently, the tire will tend to wobble, and a periodic torque must be exerted by the axle of the car to keep it rolling straight. At certain speeds, this periodic torque may excite a resonant wobbling frequency, and the tire may begin to wobble much more violently, vibrating the entire automobile.

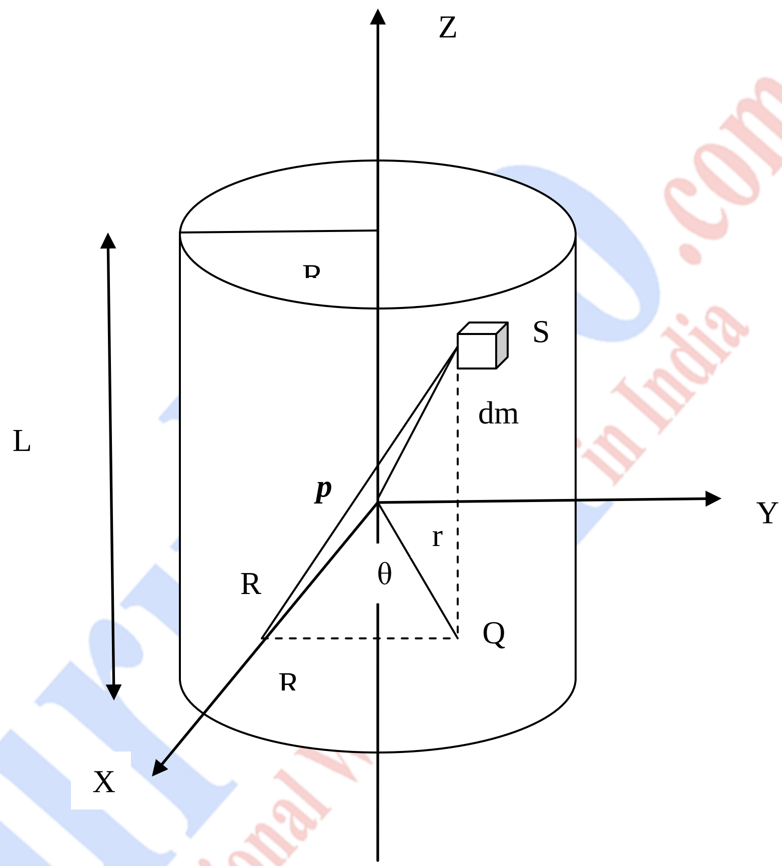
Q.3 Find the moment of inertia of a cylinder about an axis passing through its centre of mass and perpendicular to its length.

Ans Let us consider one of the principal axes along the axis of rotation of the cylinder. Origin can be taken as coinciding with the center of mass of the cylinder. Other two principal axes (co-ordinate axes) can be taken in any two perpendicular directions in a plane normal to the axis of cylinder and passing through the center of mass of the cylinder.
Let us consider R is the radius and L is the length of the cylinder and have mass M with uniform density ρ.

In cylindrical co-ordinates

x = rcosθ, y = rsinθ, z=z, where θ is measured with respect to x-axis.

The volume of this element is given by = $rd\theta . dr . dz$.



Moment of inertia of a cylinder about an axis passing through its centre of mass and perpendicular to its length passing through its centre of mass (I_X or I_y) can be obtained by

$$I_X = \int_V p^2 \rho dV$$

Where p is the distance of volume element from X- axis.

Hence

$$p^2 = z^2 + r^2 sin^2\theta$$

$$I_X = \int_0^R \int_0^{2\pi} \int_{-L/2}^{+L/2} \rho\,(z^2 + r^2 sin^2\theta) r\mathrm{dr}.\, d\theta.\, \mathrm{dz}$$

$$I_X = \rho \int_0^R \int_0^{2\pi} \left[\frac{z^3}{3} + zr^2 sin^2\theta\right]_{-L/2}^{+L/2} r\mathrm{dr}.\, d\theta$$

$$I_X = \rho \int_0^R \int_0^{2\pi} \left(\frac{L^3}{12} + Lr^2 sin^2\theta\right) r\mathrm{dr}.\, d\theta$$

$$I_X = \rho L \int_0^R \int_0^{2\pi} \left(\frac{L^2 r}{12} + r^3 sin^2\theta\right) \mathrm{dr}.\, d\theta$$

$$I_X = \rho L \int_0^{2\pi} \left[\frac{L^2}{12}\frac{r^2}{2} + \frac{r^4}{4} sin^2\theta\right]_0^R d\theta$$

$$I_X = \rho L \int_0^{2\pi} \left(\frac{L^2}{12}\frac{R^2}{2} + \frac{R^4}{4} sin^2\theta\right) d\theta$$

$$I_X = \rho L R^2 \int_0^{2\pi} \left(\frac{L^2}{24} + \frac{R^2}{4} sin^2\theta\right) d\theta$$

$$I_X = \pi \rho L R^2 \left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

$$I_X = M\left[\frac{R^2}{4} + \frac{L^2}{12}\right] = I_Y$$

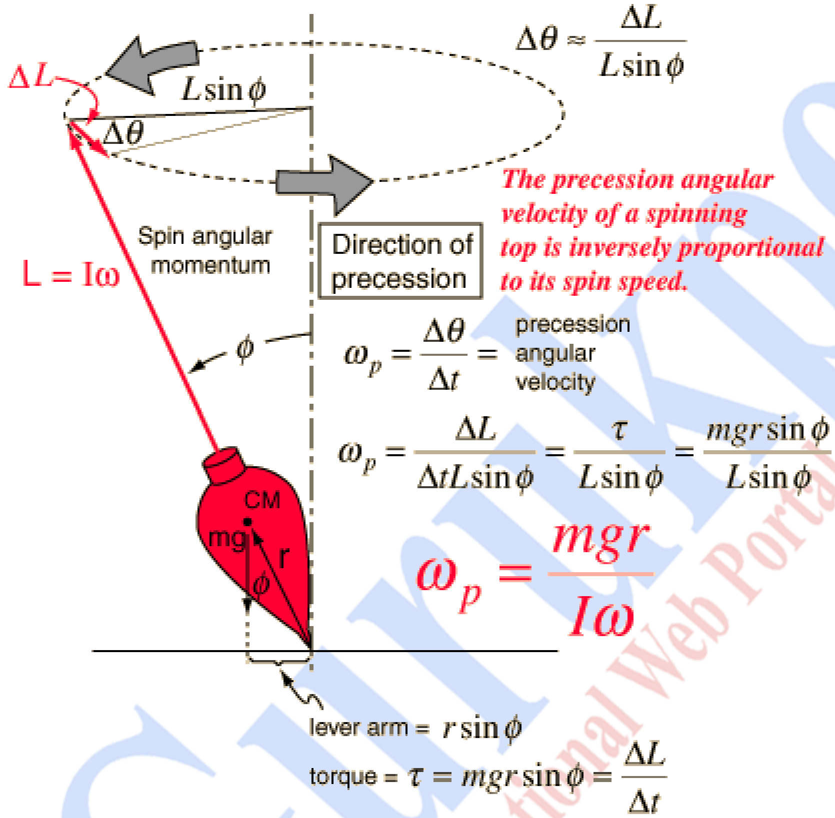
Where $M = \pi \rho L R^2$

This is the expression for moment of inertia of a cylinder about an axis passing through its centre of mass and perpendicular to its length.

Q.4 Describe precessional motion of a spinning top and calculate its precessional frequency. Explain with the suitable diagram that the rate

precession is inversely proportional to the angular momentum of a spinning top.

Ans A symmetrical object rotating about an axis which is fixed at one point is called a top. The axis of rotation of a top can itself rotate about the fixed point. The angular velocity of the axis of rotation of the top, called the precessional angular velocity is assumed to be negligible as compared to the angular velocity of the top about its axis.



The above figure represent a top spinning about its axis of symmetry with an angular momentum **J**. The axis of rotation makes angel θ at the fixed point of pivot with the vertical. The upward force at the pivot exerts no torque. The resultant torque about fixed point is due to weight mg acting through the centre of mass and is given by

$$\vec{\tau} = \boldsymbol{r} \, X \, m\boldsymbol{g}$$

Where **r** is the position vector of the centre of mass with respect to pivot O. In a short time Δt, torque produces a change in angular momentum Δ**J** given by

$$\Delta \boldsymbol{J} = \vec{\tau}\Delta t$$

The direction of the torque $\vec{\boldsymbol{\tau}}$ and change in angular momentum $\Delta\boldsymbol{J}$ is perpendicular to the axis of rotation.

The angular momentum of the top after a time Δt is the vector sum of $\boldsymbol{J}$ and $\Delta\boldsymbol{J}$ i.e. $\boldsymbol{J} + \Delta\boldsymbol{J}$**.** Since $\Delta\boldsymbol{J}$ is perpendicular to **J** and is assumed very small in magnitude compared to it, the new angular momentum vector has the same magnitude as the old one but in a different direction. Hence as the times goes on, top swings around the arc of a horizontal circle.

Hence the angular momentum vectors lies along the axis of the top, the axis turns also about a vertical line through the fixed point. This motion of rotation is called precession.

The angular velocity of precession ω_P is

$$\omega_P = \lim_{\Delta t \to 0} \frac{\Delta\emptyset}{\Delta t}$$

As ΔJ >>J, so

$$\Delta\emptyset = \frac{\Delta J}{J\sin\theta} = \frac{\tau\,\Delta t}{J\sin\theta}$$

Thus

$$\omega_P = \lim_{\Delta t \to 0} \frac{\Delta\emptyset}{\Delta t} = \frac{\tau}{J\sin\theta}$$

This equation gives precessional velocity in terms of torque τ and angular momentum J.

Hence

$$\tau = \omega_P J\sin\theta$$

Or in vector form it can be written as

$$\vec{\tau} = \overrightarrow{\omega_P}\, X\, \boldsymbol{J}$$

and as from equation $\vec{\boldsymbol{\tau}} = \boldsymbol{r}\ X\ m\boldsymbol{g} = mgr\sin\theta$

Hence

$$\omega_P = \frac{mgr\sin\theta}{J\sin\theta} = \frac{mgr}{J}$$

Thus the angular precession velocity is independent of θ and is inversely proportional to the magnitude of the angular momentum.

Q.5 Describe the moment of inertia and radius of gyration of a rigid body.

Ans **Moment of inertia:** Moment of inertia is the name given to rotational inertia, the rotational analog of mass for linear motion. It appears in the relationships for the dynamics of rotational motion. **Moment of inertia** is a property of a body that defines its resistance to a change in angular velocity about an axis of rotation. The tendency of a body to resist angular acceleration, expressed as the sum of the products of the mass of each particle in the body and the square of its perpendicular distance from the axis of rotation.

Radius of gyration: the radius of gyration of a body about an axis of rotation is defined as the radial distance of a point from the axis of rotation at which, if the whole mass of the body is assumed to be concentrated, its moment of inertia about the given axis would be the same as with its actual distribution of mass

If m is the mass of the body, its moment of inertia I in terms of its radius of gyration 'K' can be written as :

$$I = mK^2$$

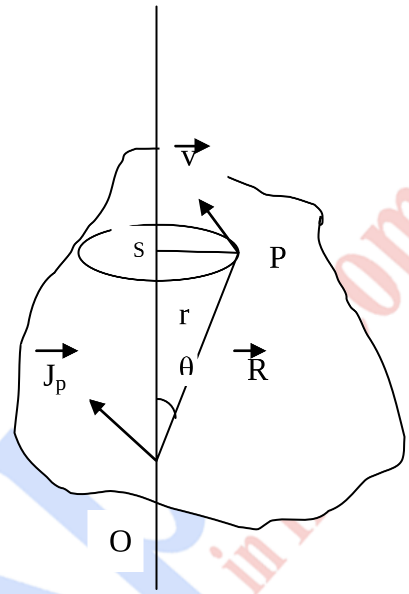
OR

$$K = \sqrt{\frac{I}{m}}$$

Physical significance: the total mass of a rotating body may be supposed to be concentrated at a radial distance 'k' from the axis of rotation so far as the moment of inertia of the body about that axis is concerned

Q.6 What is a rigid body? Find the expression for the rotational motion of a rigid body.

Ans A rigid body is made up of a number of particles such that the relative distances of the constituent particles remain unaffected under the action of a force. A rigid body is an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it. Even though such an object cannot physically exist due to relativity, objects can normally be assumed to be perfectly rigid if they are not moving near the speed of light. Find the moment of inertia of a solid sphere about its diameter.



Equation of motion of a rotating rigid body: Let us consider a rigid body rotating with an angular velocity $\vec{\omega}$ about a fixed axis OK passing through the point O. Each of constituent particle of the body moves in a circular path having its centre on the axis of rotation. If **R** is the position vector of any particle P of mass m relative to O, then its linear velocity is given by $= \vec{\omega}\ \boldsymbol{X}\ \boldsymbol{R}$
.

It is tangential to the circular path described by the particle P and perpendicular to the radius vector **R** and **v**.

Since m is the mass of the particle at P, its linear momentum is m**v** and hence its angular momentum about the point O is

$$\boldsymbol{J}_p = \boldsymbol{R}\ X\ m\boldsymbol{v}$$

Therefore the total angular moment **J** of the whole rigid body, being the vector sum of the angular momentum of all constituent particles, is obtained as

$$\boldsymbol{J} = \sum \boldsymbol{J}_p = \sum \boldsymbol{R}\ X\ m\boldsymbol{v} = \sum m\boldsymbol{R}\ X\ \boldsymbol{v} = \sum m\boldsymbol{R}\ X\ (\vec{\omega}\ \boldsymbol{X}\ \boldsymbol{R})$$

The direction of J will in general not coincide with the direction of $\vec{\omega}$ **.**

$$\boldsymbol{J} = \sum m\boldsymbol{R}\, X\, (\vec{\omega}\, \boldsymbol{X}\, \boldsymbol{R}) \; = \sum m\{\boldsymbol{R}^2\vec{\omega} - \; (\boldsymbol{R}.\vec{\omega})\boldsymbol{R} \;\}$$

If now the angel SOP = θ, then component of **R** along the axis OK is R cosθ and hence the component of **J** along this axis is of magnitude

$$J_0 = \sum m\,\{R^2\omega - (R\omega cos\theta)Rcos\theta\}$$

$$J_0 = \sum m\,\{R^2\omega(1 - cos^2\theta)\}$$

$$J_0 = \sum mR^2\omega\, sin^2\theta = \; \omega \sum mr^2$$

Where r = R sinθ is the distance of particle P from the axis of rotation.

If the moment of inertia of the rigid body about the given axis of rotation is

$$I = \sum mr^2$$

Then

$$J_0 = I\,\omega$$

i.e. the component of angular momentum along the axis of rotation is the product of I and ω.

Now, if we suppose that the axis of rotation is also the axis of symmetry of the body, the total angular momentum **J** of the body is in the same direction as axis of rotation, because the component of $\boldsymbol{J_p}$ perpendicular to OK will be cancelled out by the similar component of aangular momentum of a particle on opposite side of the circle of the radius SP. In this case

$$\boldsymbol{J} = I\,\vec{\boldsymbol{\omega}}$$

Thus the angular momentum about the axis of rotation is equal to the product of moment of inertia about that axis and angular axis.

The moment of force about O or the Torque is given by

$$\tau = \frac{d\boldsymbol{J}}{dt} = \frac{d\,(I\,\vec{\boldsymbol{\omega}})}{dt} = I\,\frac{d\vec{\boldsymbol{\omega}}}{dt} = I\,\vec{\boldsymbol{\alpha}}$$

This is the fundamental equation of motion of rigid body. If the axis of rotation is not the axis of symmetry of the body, the total angular momentum **J** of the body will not be the same as direction of $\vec{\boldsymbol{\omega}}$**.**
Taking then the components along the axis of rotation, we get

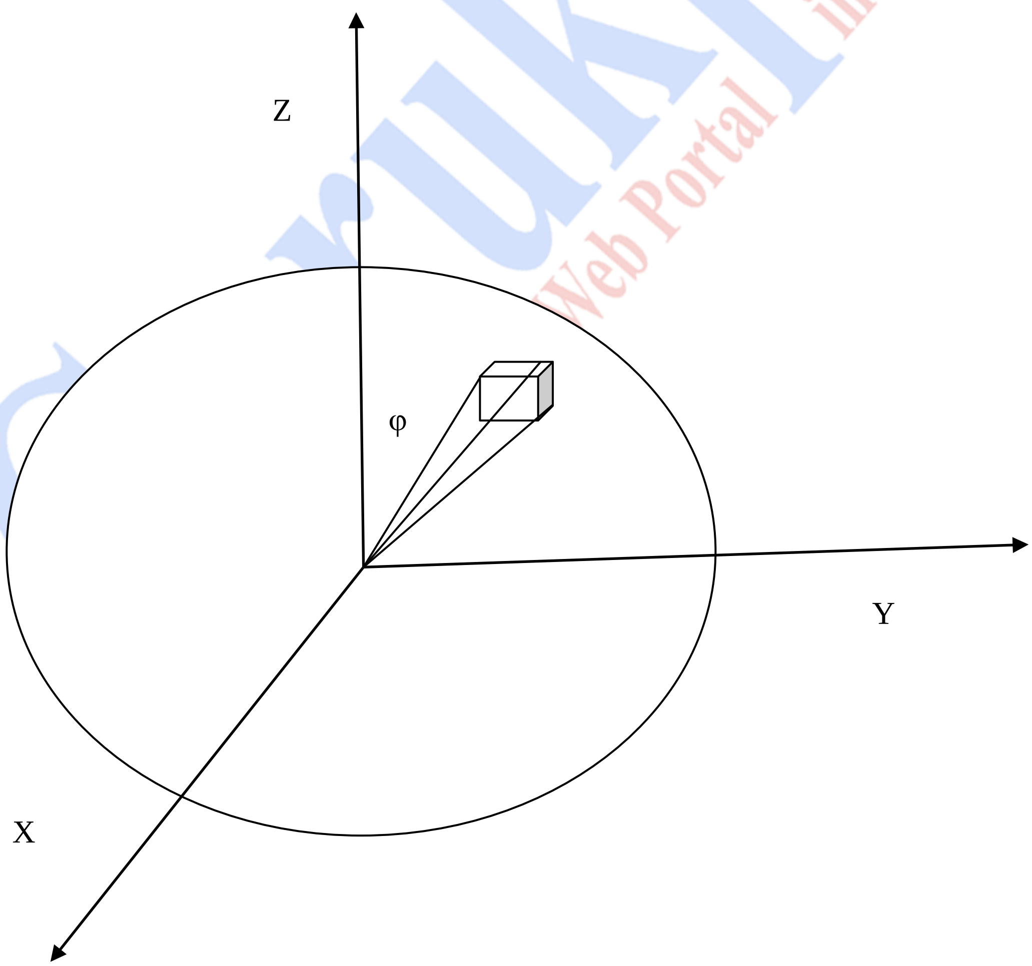
$$\tau = \frac{dJ_0}{dt} = \frac{d\,(I\,\omega)}{dt} = I\,\frac{d\omega}{dt} = I\,\alpha$$

This is the scalar form of equation of motion of rigid body about a fixed axis.

Q.7 Find the moment of inertia of a solid sphere about its diameter.

Ans The sphere is completely symmetrical in all directions about their center, hence the principal axes can be chosen in any their mutually perpendicular directions and passing through its centre. The volume density of sphere ρ is to be taken constant. The volume element at any point at a distance r from the centre is given by

$$dV = dr.rd\theta.rsin\theta.d\varphi = r^2 sin\theta.drd\theta d\varphi$$



The distance of this mass from the z axis is R sin θ. Hence the moment of inertia about z-axis is

$I_Z = \int dm(r\ sin\ \theta)^2$

$$I_Z = \int \rho\, r^2 sin\theta. drd\theta d\varphi\ (r\ sin\ \theta)^2$$

$$I_Z = \int_{r=0}^{r=R} \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=2\pi} \rho\, r^4 sin^3\theta drd\theta d\varphi$$

$$I_Z = 2\pi\rho \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} r^4 sin^3\theta drd\theta$$

$$I_Z = 2\pi\rho \frac{R^5}{5} \int_{\theta=0}^{\theta=2\pi} sin^3\theta d\theta$$

$$I_Z = 2\pi\rho \frac{R^5}{5} \int_{\theta=0}^{\theta=2\pi} \frac{1}{4}(3sin\ \theta - sin3\theta)d\theta$$

$$I_Z = 2\pi\rho \frac{R^5}{5} [-3\ cos + 3cos\theta]_0^{2\pi}$$

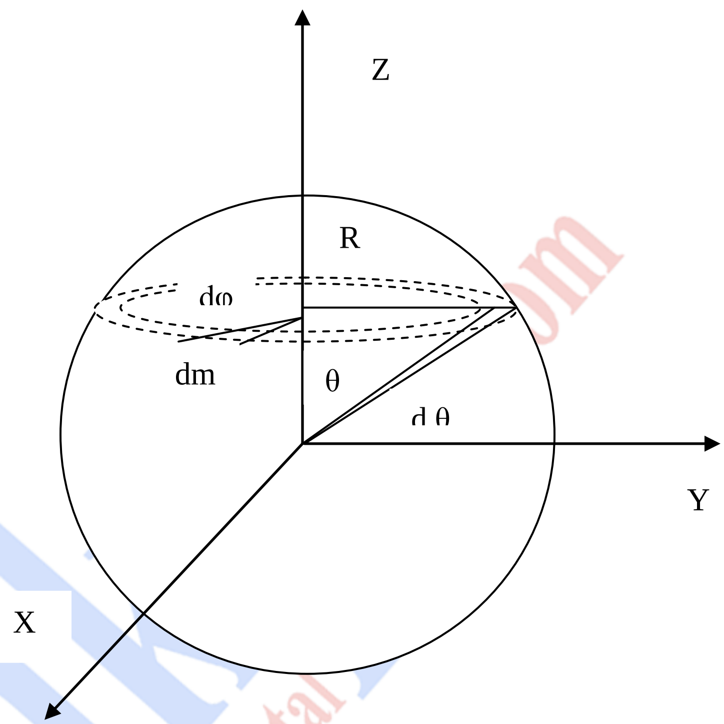
$$I_Z = \frac{2}{5}\left(\frac{4}{3}\pi R^3 \rho\right) R^2$$

$$I_Z = \frac{2}{5}\ MR^2$$

This equation gives the moment of inertia of the solid sphere about its diameter.

Q.8 Find the moment of inertia of a thin spherical shell about its diameter.

Ans The sphere is completely symmetrical in all directions about their center, hence the principal axes can be chosen in any their mutually perpendicular directions and passing through its centre. Now let us consider a infinitesimal thin shell of uniform density ρ (surface density).
Hence the total mass of the shell $M = 4\pi R^2 \rho$



And the mass of the small element lying between angel θ and θ + dθ and φ and φ + dφ is given by

$$dm = \rho\, R^2 Sin\,\theta\,.d\theta.d\varphi$$

The distance of this mass from the z axis is R sin θ. Hence the moment of inertia about z-axis is

$$I_Z = \int dm(RSin\,\theta)^2$$

$$I_Z = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \rho\, R^2 Sin\,\theta\,.d\theta.d\varphi\,(RSin\,\theta)^2$$

$$I_Z = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \rho\, R^4 Sin^3\,\theta\,.d\theta.d\varphi$$

$$I_Z = \rho\, R^4 \int_{\theta=0}^{\theta=\pi} Sin^3\,\theta\,.d\theta \int_{\varphi=0}^{\varphi=2\pi} d\varphi$$

$$I_Z = 2\pi\rho R^4 \int_{\theta=0}^{\theta=\pi} Sin^3\,\theta\,.d\theta$$

$$I_Z = 2\pi\rho R^4 \int_{\theta=0}^{\theta=\pi} \frac{1}{4}(3sin\,\theta - sin3\theta)\,d\theta$$

$$I_Z = \frac{1}{2}\pi\rho R^4\,[-3\,cos + 3cos\theta]_0^{\pi}$$

$$I_Z = \frac{1}{2}\pi\rho R^4.\frac{16}{3}$$

$$I_Z = \frac{2}{3}\,(4\pi R^2\rho)R^2$$

$$I_Z = \frac{2}{3}\,MR^2$$

Q.9 How will you distinguish between a solid and a hollow sphere if the mass and external radius R is same?

Ans The acceleration of a body rolling down an inclined plane is given by

$$a = \frac{g\,sin\theta}{1 + {k^2}/{R^2}}$$

Where k is the radius of gyration of the body with radius R.
Now for a solid sphere if k_1 is the radius of gyration, then its moment of inertia is given by

$$I = Mk_1^2 = \frac{2}{5}\,MR^2$$

Hence

$$k_1^2 = \frac{2}{5}\,R^2$$

Where R is the radius of the sphere.

Now if R and r be the external and internal radii of the hollow sphere, its radius of gyration k_2 is given by

$$I = Mk_2^2 = \frac{2}{5}\,M\left[\frac{R^5 - r^5}{R^3 - r^3}\right]$$

Or

$$k_2^2 = \frac{2}{5}\left[\frac{R^5 - r^5}{R^3 - r^3}\right]$$

Now Since R>r,

This implies $\frac{r^5}{R^5} < \frac{r^3}{R^3}$

$$\Rightarrow 1 - \frac{r^5}{R^5} > 1 - \frac{r^3}{R^3}$$

$$\Rightarrow \frac{1 - \frac{r^5}{R^5}}{1 - \frac{r^3}{R^3}} > 1$$

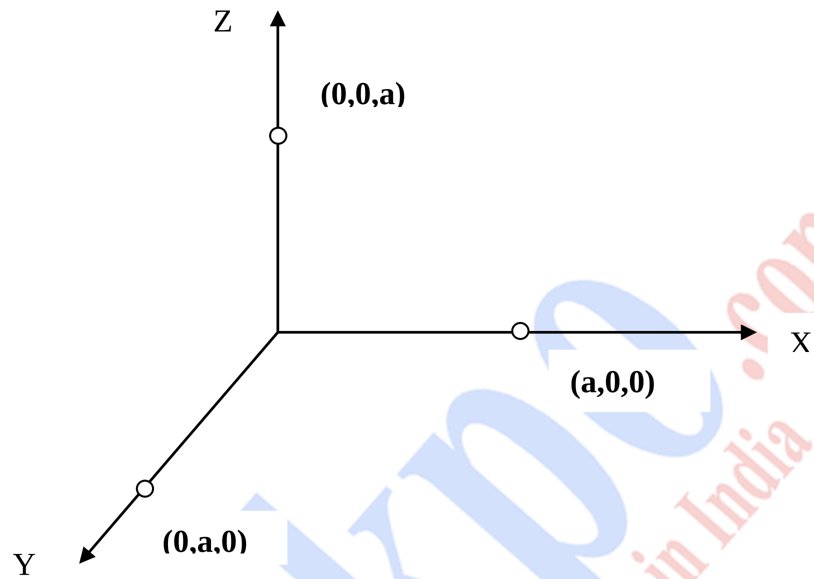
$$\Rightarrow \frac{2}{5}R^2\left(\frac{1 - \frac{r^5}{R^5}}{1 - \frac{r^3}{R^3}}\right) > \frac{2}{5}R^2$$

$$\Rightarrow k_2^2 > k_1^2$$

$$\Rightarrow k_2 > k_1$$

Hence it is clear that the radius of gyration of hollow sphere is greater than that for the solid sphere. Hence it can be observed from the expression for the acceleration of a body rolling down an inclined plane for hollow sphere is less than that for the solid sphere. It means that if both the sphere starts rolling down on an inclined plane simultaneously, the solid sphere will reach the bottom earlier. Thus by allowing the two spheres to roll down on an inclined plane, we can identify the solid and hollow spheres.

Q.10 In a system of three particles of same mass m are placed at (a,0,0), (0,a,0) and (0,0,a) respectively. Calculate all inertial coefficients of the system.



Ans All the inertial coefficient is given by

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2),\; I_{yy} = \sum_i m_i(r_i^2 - y_i^2), I_{zz} = \sum_i m_i(r_i^2 - z_i^2)$$

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i , I_{xz} = I_{zx} = -\sum_i m_i x_i z_i , I_{yz} = I_{zy} = -\sum_i m_i y_i z_i$$

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2) = m\,\{(a^2 - a^2) + (a^2 - 0^2) + (a^2 - 0^2)\} = 2ma^2$$

$$I_{yy} = \sum_i m_i(r_i^2 - y_i^2) = m\,\{(a^2 - 0^2) + (a^2 - a^2) + (a^2 - 0^2)\} = 2ma^2$$

$$I_{zz} = \sum_i m_i(r_i^2 - z_i^2) = m\,\{(a^2 - 0^2) + (a^2 - 0^2) + (a^2 - a^2)\} = 2ma^2$$

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i = -\{m(a)(0) + m(0)(a) + m(0)(0)\} = 0$$

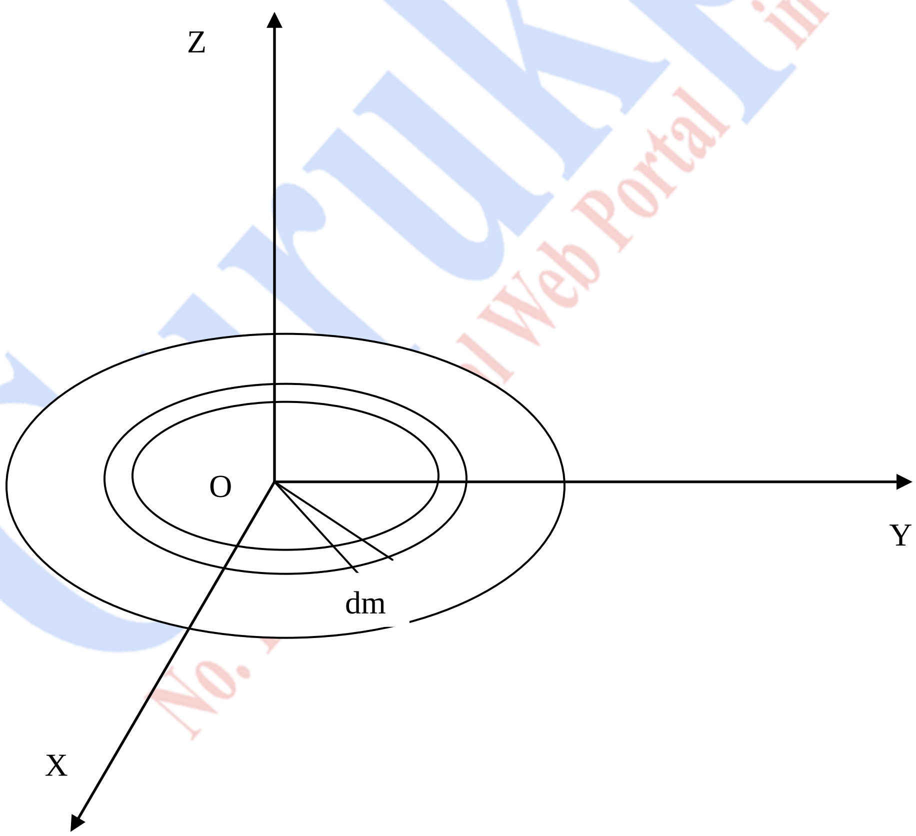
$$I_{xz} = I_{zx} = -\sum_i m_i x_i z_i = -\{m(a)(0) + m(0)(0) + m(0)(a)\} = 0$$

$$I_{yz} = I_{zy} = -\sum_i m_i y_i z_i = -\{m0(0) + m(0)(0) + m(0)(a)\} = 0$$

Q.11 Derive an expression for the moment of inertia of a disc about an axis passing through its centre of mass and perpendicular to its plane.

Ans Let us consider one of the principal axes along the axis of rotation of the disc i.e. an axis passing through its centre of mass and perpendicular to its plane. Let z-axis be chosen along this direction, the other two will lie in a plane of lamina. Let one such element lie between angle θ and θ + d θ and at a distance r and r+dr from the centre.

The volume of this element is given by $dV = t.rd\theta.\mathrm{dr}$



Moment of Inertia about z-axis is given by

$$I_Z = \int_0^R \int_0^{2\pi} \rho r^2 dV = \int_0^R \int_0^{2\pi} \rho r^2 t. rd\theta. \mathrm{dr}$$

$$I_Z = \int_0^R \int_0^{2\pi} t\rho r^3 \mathrm{dr} d\theta$$

$$I_Z = 2\pi t\rho \int_0^R r^3 \mathrm{dr}$$

$$I_Z = 2\pi t\rho \frac{R^4}{4} = \frac{\pi t\rho R^4}{2}$$

But mass of the disc = Volume X Density $=\pi R^2 t$

Hence

$$I_Z = \frac{1}{2} M R^2$$

Q.12 Express the rotational kinetic energy of rigid body in terms of inertial coefficients and angular velocity components.

Ans In rotational motion, kinetic energy of rigid body is given by:

$$K_{rot} = \sum_i \frac{1}{2} m_i {v_i}^2$$

Where m_i is the mass of the i[th] particle of the body moving with a linear velocity $\vec{v_i}$. If $\vec{\omega}$ is the angular velocity of the body, then

$$\vec{v_i} = \vec{\omega} \,\mathrm{X}\, \vec{r_i}$$

$${v_i}^2 = \vec{v_i}\,.\vec{v_i} = (\vec{\omega}\,\mathrm{X}\,\vec{r_i}\,).(\vec{\omega}\,\mathrm{X}\,\vec{r_i}\,) = \omega^2 {r_i}^2 - (\vec{\omega}\,.\vec{r_i}\,)^2$$

Angular velocity vector $\vec{\omega}$ and the position vector $\vec{r_i}$ of the i[th] particle can be resolved into three mutually perpendicular components along the axes of Cartesian coordinate system. Thus, we have

$$\vec{\omega} = \omega_x\,\hat{\imath} + \omega_y\,\hat{\jmath} + \omega_z\,\hat{k}$$

$$\vec{r} = x_i\,\hat{\imath} + y_i\,\hat{\jmath} + z_i\,\hat{k}$$

$$\omega^2 = \vec{\omega}.\vec{\omega} = {\omega_x}^2 + {\omega_y}^2 + {\omega_z}^2$$

And

$$r_i^2 = \vec{r_i}.\vec{r_i} = x_i^2 + y_i^2 + z_i^2$$
$$\vec{\omega}.\vec{r_i} = \omega_x x_i + \omega_y y_i + \omega_z z_i$$
$$(\vec{\omega}.\vec{r_i})^2 = \omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2 + 2\,\omega_x\omega_y x_i y_i + 2\omega_y\omega_z y_i z_i + 2\omega_z\omega_x z_i x_i$$

On substituting these values we get

$$v_i^2 = \omega^2 r_i^2 - (\vec{\omega}.\vec{r_i})^2$$
$$v_i^2 = (\omega_x^2 + \omega_y^2 + \omega_z^2)(x_i^2 + y_i^2 + z_i^2) - \omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2 + 2\,\omega_x\omega_y x_i y_i + 2\omega_y\omega_z y_i z_i + 2\omega_z\omega_x z_i x_i$$
$$v_i^2 = \omega_x^2 (r_i^2 - x_i^2) + \omega_y^2 (r_i^2 - y_i^2) + \omega_z^2 (r_i^2 - z_i^2) - 2\,\omega_x\omega_y x_i y_i + 2\omega_y\omega_z y_i z_i + 2\omega_z\omega_x z_i x_i$$

Therefore,

$$K_{rot} = \sum_i \frac{1}{2} m_i v_i^2$$
$$= \sum_i \frac{1}{2} m_i\,[\omega_x^2 (r_i^2 - x_i^2) + \omega_y^2 (r_i^2 - y_i^2) + \omega_z^2 (r_i^2 - z_i^2) - 2\,\omega_x\omega_y x_i y_i + 2\omega_y\omega_z y_i z_i + 2\omega_z\omega_x z_i x_i]$$

We know that the inertial coefficients are given by:

$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2)$$
$$I_{yy} = \sum_i m_i(r_i^2 - y_i^2)$$
$$I_{zz} = \sum_i m_i(r_i^2 - z_i^2)$$
$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i$$
$$I_{xz} = I_{zx} = -\sum_i m_i x_i z_i$$
$$I_{yz} = I_{zy} = -\sum_i m_i y_i z_i$$

Hence

$$K_{rot} = \frac{1}{2}\,[I_{xx}{\omega_x}^2 + I_{YY}{\omega_y}^2 + \ I_{ZZ}{\omega_z}^2 + 2\ \omega_x\omega_y I_{XY}\ + 2\ \omega_y\omega_z I_{YZ} +\ +2\ \omega_z\omega_x I_{ZX}]$$

Which can also be written as

$$K_{rot} = \frac{1}{2}\,\vec{L}.\vec{\omega} = \frac{1}{2}\,(I\,\vec{\omega}\,).\vec{\omega} = \frac{1}{2}\,I\omega^2 = \frac{L^2}{2I}$$

This gives the rotational kinetic energy of the rotating rigid body.

Multiple Choice Questions

1. If radius of solid sphere is doubled by keeping its mass constant, then

(a) $\frac{I_1}{I_2} = \frac{1}{4}$

(b) $\frac{I_1}{I_2} = \frac{4}{1}$

(c) $\frac{I_1}{I_2} = \frac{3}{2}$

(d) $\frac{I_1}{I_2} = \frac{2}{3}$

Answer: (a)

2. Calculate the M.I. of a thin uniform ring about an axis tangent to the ring and in a plane of the ring, if its M.I. about an axis passing through the centre and perpendicular to plane is 4 kg m^2.

(a) 12 kg m^2

(b) 3 kg m^2

(c) 6 kg m^2

(d) 9 kg m^2

Answer: (c)

3. A thin uniform ring of mass M and radius R passing through its centre and perpendicular to its plane. Then its M.I. is, 82)

(a) $\frac{1}{2}MR^2$

(b) MR^2

(c) $2MR^2$

(d) $\frac{3}{2}MR^2$

Answer: (b)

4. Two bodies have their moments of inertia I and 2I respectively about their axis of rotation. If their kinetic energies of rotation are equal, their angular momenta will be in the ratio of

(a) 1:2

(b) 2:1

(c) $\sqrt{2}:1$

(d) $1:\sqrt{2}$

Answer: (d)

6. Radius of gyration of disc rotating about an axis perpendicular to its passing through its centre is

(a) $\frac{R}{2}$

(b) $\frac{R}{\sqrt{2}}$

(c) $\frac{R}{\sqrt{3}}$

(d) $\frac{R}{3}$

Answer: (b)

7. The moment of inertia of a body comes into play

(a) In motion along a curved path

(b) In linear motion

(c) In rotational motion

(d) None of the above

Answer: (c)

8. A solid sphere, a hollow sphere and a disc are released from the top of a frictionless inclined plane so that they slide down the inclined plane (without rolling). The maximum acceleration down the plane is

(a) For the solid sphere
(b) For the hollow sphere
(c) For the disc
(d) The same for all bodies

Answer: (d)

9. Moment of inertia depends on

(a) Distribution of particles
(b) Mass
(c) Position of axis of rotation
(d) All of these

Answer: (d)

10. The moment of inertia of an electron in n^{th} orbit will be

(a) MR_n^2

(b) $\frac{MR_n^2}{2}$

(c) $\frac{1}{2}MR_n^2$

(d) $\frac{2}{3}MR_n^2$

Answer: (a)

11. The M.I. of a body does not depends upon

(a) Angular velocity of a body
(b) Axis of rotation of the body
(c) The mass of the body
(d) The distribution of the body

Answer: (a)

Unit 4

Motion Under Central Forces

Q.1 Explain the conservation of angular momentum and deduce the Kepler's second law of planetary motion.

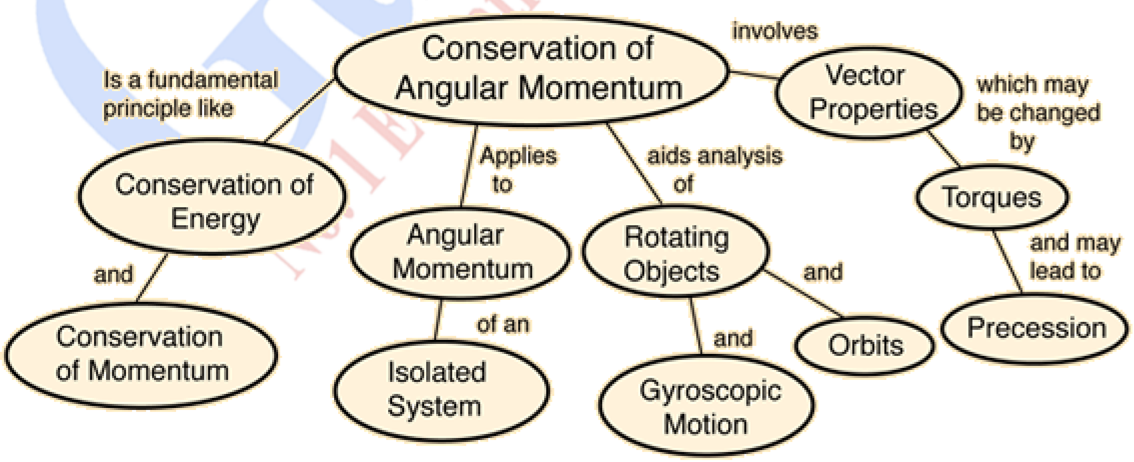
Ans Conservation of angular momentum:

We know that the external torque applied to a system of particles is given by

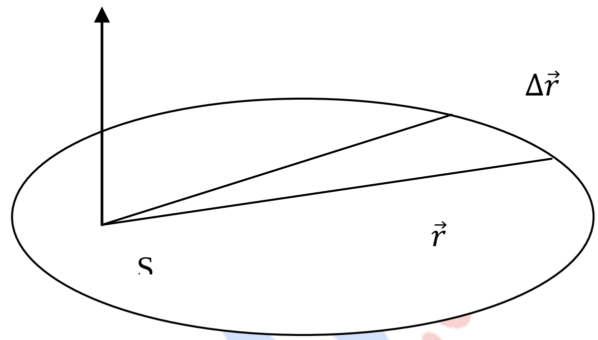
$$\vec{\tau} = \frac{d\vec{L}}{dl}$$

Now if $\vec{\tau} = 0$, then $\frac{d\vec{L}}{dl} = 0, \vec{L} = constant$

i.e. when the external torque applied to the system is zero, the total angular momentum of the system remains conserved. This is the law of conservation of angular momentum.



Kepler's second law: The radius vector joining sun and the planet sweeps equal area in equal times i.e. areal velocity of the planet is always constant.



The area swept by planet in time Δt is given by

$$\Delta\vec{A} = \frac{1}{2}\,\vec{r}X\Delta\vec{r}$$

Hence the areal velocity of the planet is

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\,\vec{r}X\frac{\Delta\vec{r}}{\Delta t}$$

Now

$$\lim_{\Delta t--0}\frac{\Delta\vec{A}}{\Delta t} = \frac{d\vec{A}}{dt} = \frac{1}{2}\,\vec{r}X\frac{d\vec{r}}{dt} = \frac{1}{2\mu}\,\vec{r}X\,\mu\,\vec{v} = \frac{1}{2\mu}\,\vec{r}\,X\,\vec{P} = \frac{\vec{L}}{2\mu}$$

Where μ is the reduced mass of the system, $\vec{P}$ is linear momentum and $\vec{L}$ is the angular momentum.

Since the angular momentum $\vec{L}$ in central force for isolated system is constant, hence the areal velocity must also be constant. Thus

$$\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2\mu} = Constant$$

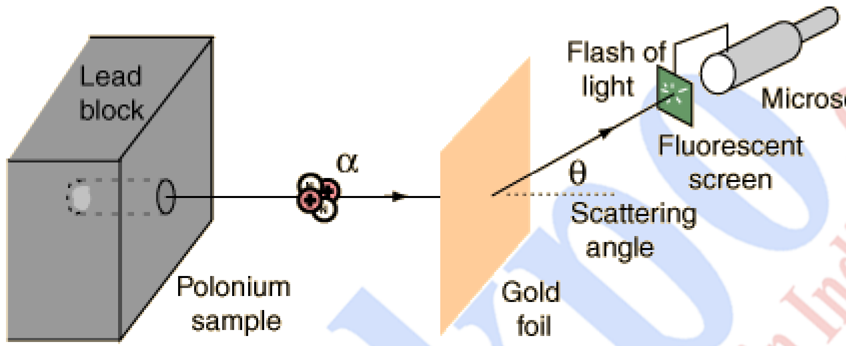
Q.2 Define Rutherford scattering. Deduce the relation between scattering angle and impact parameter of α scattering.

or

Calculate the distance of closest approach of a proton of energy V electron volt, if it moves towards a nucleus of atomic number z.

Rutherford Scattering

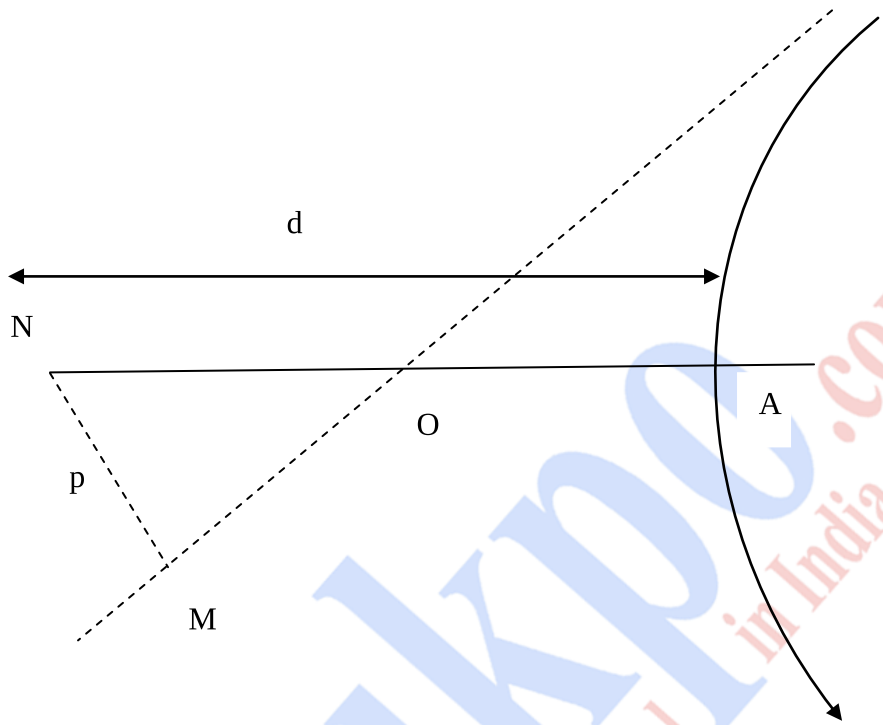
Ans Alpha particles from a radioactive source were allowed to strike a thin gold foil. Alpha particles produce a tiny, but visible flash of light when they strike a fluorescent screen. Surprisingly, alpha particles were found at large deflection angles and some were even found to be back-scattered.



This experiment showed that the positive matter in atoms was concentrated in an incredibly small volume and gave birth to the idea of the nuclear atom. In so doing, it represented one of the great turning points in our understanding of nature.

Rutherford Scattering Formula

Let us consider the nucleus N of charge Ze is very heavy and stationary and a light positively charged particle of charge ne moving along PO approaches to it. Both the nucleus and the charged particle are positively charged and there will be a force of repulsion between them given by coulomb's law. This repulsive force will go on increasing as the positively charged particle gets closer to nucleus.



The charged particle of initial velocity v_0 is repelled by the heavy positive nucleus and changes from straight line to a hyperbola PAQ having one focus at N. The asymptotes PO and OQ give the initial and final directions of the charged particles. The perpendicular distance of OQ from N = MN= p. This is the shortest distance from nucleus to the initial direction of motion of the charged particle. This distance is called impact parameter.

If m is the mass of the charged particle, the initial angular momentum of the charged particle about heavy nucleus N = m v_0p

At the distance of closest approach i.e. at A the angular momentum = mv(NA)= mvd, where v is the velocity of the charged particle at point A.
Since here torque $\tau = \boldsymbol{r}\ X\ \boldsymbol{F} = \boldsymbol{r}\ X\ K\ \hat{\boldsymbol{r}} = 0,$ i.e. angular momentum is conserved. Hence according to conservation of angular momentu, we get

$$mv_0p = mvd$$

$$v = \frac{v_0p}{d}$$

The initial energy of the charged particle is wholly kinetic energy and is equal to $\frac{1}{2}m{v_0}^2$

Hence the total energy at point at A $= \frac{1}{2}mv^2 + \frac{1}{4\pi\epsilon_0}\frac{(Ze)(ne)}{d}$

Now according to the law of conservation of energy, we get

$$\frac{1}{2}m{v_0}^2 = \frac{1}{2}mv^2 + \frac{1}{4\pi\epsilon_0}\frac{(Ze)(ne)}{d}$$

$$\frac{1}{4\pi\epsilon_0}\frac{(Ze)(ne)}{d} = \frac{1}{2}m{v_0}^2 - \frac{1}{2}mv^2$$

$$\frac{1}{4\pi\epsilon_0}\frac{(Ze)(ne)}{d} = \frac{1}{2}m{v_0}^2 - \frac{1}{2}m\left(\frac{v_0 p}{d}\right)^2$$

$$\frac{1}{4\pi\epsilon_0}\frac{Zne^2}{d} = \frac{1}{2}m{v_0}^2\left\{1 - \left(\frac{p}{d}\right)^2\right\}$$

This equation gives the value of d i.e. distance of closest approach

For proton n = 1 and for α- particle n =2, therefore

For proton scattering,

$$\frac{1}{4\pi\epsilon_0}\frac{Ze^2}{d} = \frac{1}{2}m{v_0}^2\left\{1 - \left(\frac{p}{d}\right)^2\right\}$$

for α- particle scattering,

$$\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{d} = \frac{1}{2}m{v_0}^2\left\{1 - \left(\frac{p}{d}\right)^2\right\}$$

In case of a head on collision, the impact parameter p =0, then equation reduces to

$$\frac{1}{4\pi\epsilon_0}\frac{Zne^2}{d} = \frac{1}{2}m{v_0}^2$$

Or

$$d = \frac{1}{4\pi\epsilon_0}\frac{2Zne^2}{m{v_0}^2}$$

Q.3 Deduce the equation of motion of a particle under the influence of central force.

The motion under central forces is always confined to a plane. Let this plane be XY plane and let its position be defined by r and θ in polar co-ordinates. Then at any instant x= r cosθ, y= r sinθ. Hence the position vector of particle is

$$\vec{r} = r\, cos\theta \hat{\imath} + \ r\, sin\theta \hat{\jmath}$$

Where r and θ are the functions of time. Differentiating this equation with respect to t, we get

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dr}{dt} cos\theta - rsin\theta\ \frac{d\theta}{dt}\right)\ \hat{\imath} + \left(\frac{dr}{dt} sin\theta + rcos\theta\ \frac{d\theta}{dt}\right)\ \hat{\jmath}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\ (cos\theta\hat{\imath} + sin\theta\hat{\jmath}) + r\ \frac{d\theta}{dt}(-sin\theta\hat{\imath} + cos\theta\hat{\jmath}\ )$$

But

$(cos\theta\hat{\imath} + sin\theta\hat{\jmath}) = \ \hat{r}, the\ unit\ vector\ along\ the\ position\ vector\ of\ the\ particle$

$(-sin\theta\hat{\imath} + cos\theta\hat{\jmath}\ ) = \hat{\theta}, unit\ vector\ directed\ along\ the\ perpendicular\ to\ \vec{r}$

So

$$\vec{v} = \ \dot{r}\hat{r} + \ r\dot{\theta}\hat{\theta}$$

Thus the velocity of the particle is a vector which is sum of the two perpendicular vectors namely radial velocity and transverse velocity.

The instantaneous acceleration of the particle is obtained by differentiating above equation with respect to t again, we get

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = \left[\ddot{r}\ cos\theta - 2\dot{r}\ sin\theta\ \dot{\theta} - \ r\ cos\theta\ \left(\dot{\theta}\right)^2 - \ rsin\ \theta\ \ddot{\theta}\right]\hat{\imath}$$
$$+\left[\ddot{r}\ sin\theta + 2\dot{r}\ cos\theta\ \dot{\theta} - r\ sin\theta\ \left(\dot{\theta}\right)^2 + r\ cos\theta\ \ddot{\theta}\right]\hat{\jmath}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = \left\{\ddot{r} - \ r\ \left(\dot{\theta}\right)^2\right\}(cos\theta\hat{\imath} + sin\theta\hat{\jmath}) + \ \left(2r\dot{\theta} + r\ddot{\theta}\right)(-sin\theta\hat{\imath} + cos\theta\hat{\jmath}\ )$$

$$\vec{\ddot{r}} = \left\{\ddot{r} - r\left(\dot{\theta}\right)^2\right\}\hat{r} + \left(2r\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}$$

Thus the acceleration of the particle is a vector which is sum of the two perpendicular vectors namely radial acceleration and transverse acceleration.

Therefore the equation of motion of the system under the action of central force can be expressed as

$$\mu\left[\left\{\ddot{r} - r\left(\dot{\theta}\right)^2\right\}\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}\right] = f(r)\,\hat{r}$$

Where μ is the reduced mass of the system.

Thus, we have

$$\mu\left\{\ddot{r} - r\left(\dot{\theta}\right)^2\right\} = f(r)$$

And

$$\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0$$

Now since the angular momentum L is a constant and given by

$$\vec{L} = \mu r^2\vec{\dot{\theta}}$$

Where $\vec{L}$ and $\vec{\dot{\theta}}$ are parallel, then

$$\dot{\theta} = \frac{L}{\mu r^2}$$

Hence from the above equation, we can say that

$$\mu\left\{\ddot{r} - r\left(\frac{L}{\mu r^2}\right)^2\right\} = f(r)$$

Or

$$\ddot{r} = \frac{f(r)}{\mu} + \frac{L^2}{\mu^2 r^3}$$

And the second equation becomes

$$\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = -\frac{2}{r}\frac{dr}{dt}.\frac{d\theta}{dt} = -\frac{2}{r}\frac{dr}{d\theta}.\frac{d\theta}{dt}.\frac{d\theta}{dt} = -\frac{2}{r}\frac{dr}{d\theta}.\left(\frac{d\theta}{dt}\right)^2$$

But

$$\ddot{r} = \frac{d^2r}{dt^2} = \left(\frac{d\theta}{dt}\right)^2 \frac{d^2r}{d\theta^2} + \frac{dr}{d\theta}\frac{d^2\theta}{dt^2}$$

$$\frac{d^2r}{dt^2} = \left(\frac{d\theta}{dt}\right)^2 \frac{d^2r}{d\theta^2} + \frac{dr}{d\theta}\left\{-\frac{2}{r}\frac{dr}{dt}.\frac{d\theta}{dt}\right\}$$

$$\frac{d^2r}{dt^2} = \left(\frac{d\theta}{dt}\right)^2 \left\{\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$\frac{d^2r}{dt^2} = \left(\frac{L}{\mu r^2}\right)^2 \left\{\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\right\}$$

Put this value in the above radial acceleration equation, we get

$$\left(\frac{L}{\mu r^2}\right)^2 \left\{\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\right\} = \frac{f(r)}{\mu} + \frac{L^2}{\mu^2 r^3}$$

After rearranging the equation of motion, we get

$$\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = \frac{\mu r^4 f(r)}{L^2}$$

This equation gives the equation of path of the motion of the particle in terms of r and θ.

Q.4 A satellite has maximum and minimum orbital velocities V_{max} and V_{min}. Prove that the eccentricity of satellite is:-

$$\epsilon = \frac{V_{max} - V_{min}}{V_{max} + V_{min}}$$

$$\epsilon = \frac{V_{max} - V_{min}}{V_{max} + V_{min}}$$

Ans The velocity of a planet moving in an elliptical orbit around the sun at its turning point is given as

$$v = \frac{Gm_P m_S}{L}\,(\in \pm 1)$$

Where m_P and m_S are the masses of the planet and the sun respectively, L is the angular momentum and the -ve and +ve sign corresponds to maximum and minimum velocities of planet.

i.e.

$$V_{max} = \frac{Gm_P m_S}{L}(\in -1)$$

And

$$V_{min} = \frac{Gm_P m_S}{L}(\in +1)$$

Thus we have

$$\frac{V_{max}}{V_{min}} = \frac{(\in -1)}{(\in +1)}$$

Which gives

$$\epsilon = \frac{V_{max} - V_{min}}{V_{max} + V_{min}}$$

Q.5 Distance of two planets from the sun is 10^{13} and 10^{12} meter respectively. Find out time periods and velocities of these planets.

Ans The time period of a satellite moving in circular orbit of radius r is given by

$$T^2 \propto r^3$$

And the velocity is given by $v = r\omega = \frac{2\pi r}{T}$

Hence $v \propto \frac{1}{r^2}$

Thus

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3 \; and \; \left(\frac{v_1}{v_2}\right)^2 = \left(\frac{r_2}{r_1}\right)^2$$

i.e.

$$\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2} \; and \; \frac{v_1}{v_2} = \frac{r_2}{r_1}$$

Q.6 Explain central forces. When a particle moves under the action of central force, prove that-

(i) Its motion takes place in a plane

(ii) Its angular momentum is constant

(iii) Its areal velocity is constant

Ans

The forces which act on a particle along a line joining the centre of mass of the particle to some fixed point are called the central forces and the point is known as centre of force. The fixed point is generally taken as origin and force is a function of distance r from this point. It is generally expressed as

$$\vec{F} = f(r)\hat{r}$$

Where f(r) is some function of r, f(R) <0 for attractive forces and f(r) >0 for repulsive forces.

Hence, in central force problem, we have

$$\vec{r}X\,\vec{F} = \vec{r}Xf(r)\hat{r} = 0$$

i.e. $\tau = \frac{d\vec{L}}{dt} = \vec{r}X\,\vec{F} = 0$

Hence $\vec{L} = constant$

i.e. when a particle moves under the action of central force, its angular momentum is conserved.

Now $\vec{r}.\vec{L} = \vec{r}.\left(\vec{r}X\,\vec{P}\right) = 0$

i.e. $\vec{r}$ is perpendicular to the constant vector $\vec{L}$. Hence we can say that motion takes plane in a plane for central forces which is defined by the initial direction of motion of the particle and the initial line connecting it to the force centre known as radius vector.

In the motion under central forces, the areal velocity i.e. the rate by which the position vector of the particle sweeps the area is constant.

The area swept by planet in time Δt is given by

$$\Delta\vec{A} = \frac{1}{2}\,\vec{r}X\Delta\vec{r}$$

Hence the areal velocity of the planet is

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\,\vec{r}X\frac{\Delta\vec{r}}{\Delta t}$$

Now

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{A}}{\Delta t} = \frac{d\vec{A}}{dt} = \frac{1}{2}\,\vec{r} X \frac{d\vec{r}}{dt} = \frac{1}{2\mu}\,\vec{r} X\,\mu\,\vec{v} = \frac{1}{2\mu}\,\vec{r}\,X\,\vec{P} = \frac{\vec{L}}{2\mu}$$

Where μ is the reduced mass of the system, $\vec{P}$ is linear momentum and $\vec{L}$ is the angular momentum.

Since angular moment is constant in this motion, hence areal velocity i.e. the rate by which the position vector of the particle sweeps the area will also be constant.

Q.7 Discuss the gravitational mass and inertial mass of a body.

Ans **Gravitational mass:** Gravitational mass of the body can be defined as the mass of a body in terms of the gravitational force of attraction exerted on it by another body (eg. earth) i.e.

$$F = \frac{GMm_g}{r^2}$$

Or

$$m_g = \frac{Fr^2}{GM}$$

Here m_g is the gravitational mass and M is the mass of the earth and r is the distance of the body from the centre of the earth.

Inertial mass: Newton's first law of motion states that "Every body continues to be in its state of rest or uniform motion until and unless acted upon by the external forces". This inertness of the body to change its state of rest or of uniform motion is called inertia and is an inherent property of the body by virtue of its mass called inertial mass and can be measured as the ratio of the force applied (F) and acceleration (a) produced in the body". i.e.

$$m_i = \frac{F}{a}$$

Here m_i is the inertial mass of the body.

The gravitational mass and the inertial mass of the bodyare found to be proportional to each other.

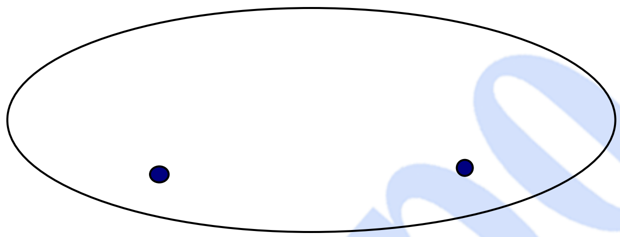
Q.8 Give Kepler's law for planetary motion and derive Kepler's area law.

Kepler's law for planetary motion:

Ans Kepler's first law:

Each planet moves in an ellipse with the Sun at one focus.

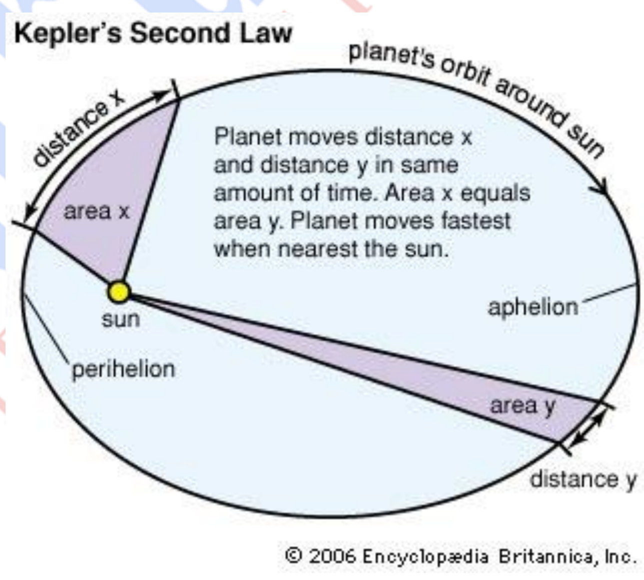
Ellipse is a closed figure drawn around two points (foci) in such a way that the sum of the distances from any point on the ellipse to the foci equals a constant.



Kepler's second law:

The line between the Sun and the planet sweeps over equal areas in equal time intervals.

Speed of a planet along an orbit is not constant: a planet moves faster when closer to the Sun and slower, when farther away.



Kepler's third law

*The ratio of the cube of the semimajor axis (**a**) to the square of the period of revolution (**P**) is the same (**K**) for each planet.*

$$\frac{a^3}{P^2} = K$$

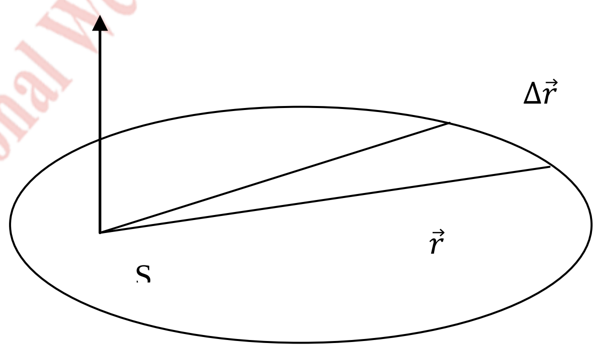
or

the ratio of the squares of the periods of any two planets revolving about the sun is equal to the ratio of the cubes of their average distances from the sun, where *T* is period, *r* is average distance from the sun, and *a* and *b* represent two planets; provided accurate data for the distance the planets are from the sun.

$$\left(\frac{T_a}{T_b}\right)^2 = \left(\frac{r_a}{r_b}\right)^3$$

The Major axis of an ellipse is its longest diameter; the semi major axis is half that.

Kepler's second law: The radius vector joining sun and the planet sweeps equal area in equal times i.e. areal velocity of the planet is always constant.



The area swept by planet in time Δt is given by

$$\Delta\vec{A} = \frac{1}{2}\,\vec{r} X \Delta\vec{r}$$

Hence the areal velocity of the planet is

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\,\vec{r} X \frac{\Delta\vec{r}}{\Delta t}$$

Now

$$\lim_{\Delta t--0}\frac{\Delta\vec{A}}{\Delta t}=\frac{d\vec{A}}{dt}=\frac{1}{2}\vec{r}X\frac{d\vec{r}}{dt}=\frac{1}{2\mu}\vec{r}X\,\mu\,\vec{v}=\frac{1}{2\mu}\vec{r}\,X\,\vec{P}=\frac{\vec{L}}{2\mu}$$

Where μ is the reduced mass of the system, $\vec{P}$ is linear momentum and $\vec{L}$ is the angular momentum.

Since the angular momentum $\vec{L}$ in central force for isolated system is constant, hence the areal velocity must also be constant. Thus

$$\frac{d\vec{A}}{dt}=\frac{\vec{L}}{2\mu}=Constant$$

Q.9 Show that $r=\frac{p}{1+\varepsilon cos\theta}$. How does the path of a moving body depend on ε in Cartesian coordinates?

Ans

$$r=\frac{p}{1+\varepsilon cos\theta}$$

In Cartesian coordinate,

$$x=r\,cos\theta, y=r\,sin\theta, r^2=x^2+y^2$$

The give n equation is

$$r\,(1+\varepsilon cos\theta)=p$$

$$r+r\varepsilon cos\theta=p$$

$$\sqrt{x^2+y^2}+\varepsilon\,x=p$$

Or

$$\sqrt{x^2+y^2}=p-\varepsilon\,x$$

On squaring both the side

$$x^2+y^2=(p-\varepsilon\,x)^2$$

$$x^2+y^2=p^2+\varepsilon^2x^2-2p\,\varepsilon x$$

Or

$$(1-\varepsilon^2)x^2+2p\,\varepsilon x+y^2=p^2$$

This is the equation of the path of the moving body.

Q.10 The average distance of Mars from earth is 1.524 times greater than the average distance of earth from sun. Find the time period of Mars revolving around Sun.

Ans The time period of a satellite moving in circular orbit of radius r is given by

$$T^2 \propto r^3$$

Now the time period for earth ${T_{earth}}^2 \propto {r_{earth}}^3$ and ${T_{Mars}}^2 \propto {r_{Mars}}^3$

$$r_{Mars}\, from\, Sun = r_{Mars}\, from\, earth + r_{earth}\, from\, sun$$

$$= 1.524\, r_{earth} + r_{earth}$$

$$r_{Mars} = 2.524\, r_{earth}$$

$$\frac{{T_{Mars}}^2}{{T_{earth}}^2} = \frac{{r_{Mars}}^3}{{r_{earth}}^3} = \frac{(2.524\, r_{earth})^3}{{r_{earth}}^3} = (2.524)^3 = 16.0793$$

$$T_{Mars} = \sqrt{16.0793}\, T_{earth} = 4.01\, T_{earth} = 4.01\,(24\, Hr) = 96\, Hr$$

Multiple Choice Questions

1. A dancer on ice spins faster when she folds here arms. This is due to

(a) Increases in energy and increase in angular momentum

(b) Decrease in friction at the skates

(c) Constant angular momentum and increase in kinetic energy

(d) Increase in energy an decreases in angular momentum

Answer: (c)

2. A mass is revolving in a circle which is in the plane of the paper. The direction of angular acceleration is

(a) Upward to the radius

(b) Towards the radius

(c) Tangential

(d) At right angle to angular velocity

Answer: (c)

3. Angular momentum is

(a) A scalar

(b) A polar vector

(c) A scalar as well as vector

(d) An axial vector

Answer: (d)

4. Dimensions of angular momentum is

(a) $[M^1L^2T^2]$

(b) $[M^1L^2T^1]$

(c) $[M^1L^2T^2]$

(d) $[M^1L^0T^1]$

Answer: (c)

5. The term moment of momentum is called

(a) Momentum

(b) Force

(c) Torque

(d) Angular momentum

Answer: (d)

6. When a steady torque is acting on a body, the body

(a) Continues in its state of rest or uniform motion along a straight line

(b) Gets linear acceleration

(c) Gets angular acceleration

(d) Rotates at a constant speed

Answer: (d)

7. Centre of mass of two body system divides the distance between two bodies, is proportional to

(a) Inverse of square of the mass

(b) Inverse of mass

(c) The ratio of the square of mass

(d) The ratio of mass

Answer: (b)

Unit 5

Elastic Properties of Matter

Q.1 Prove that the necessary torque required to twist a solid cylinder of length l and radius r by an angel θ is:-

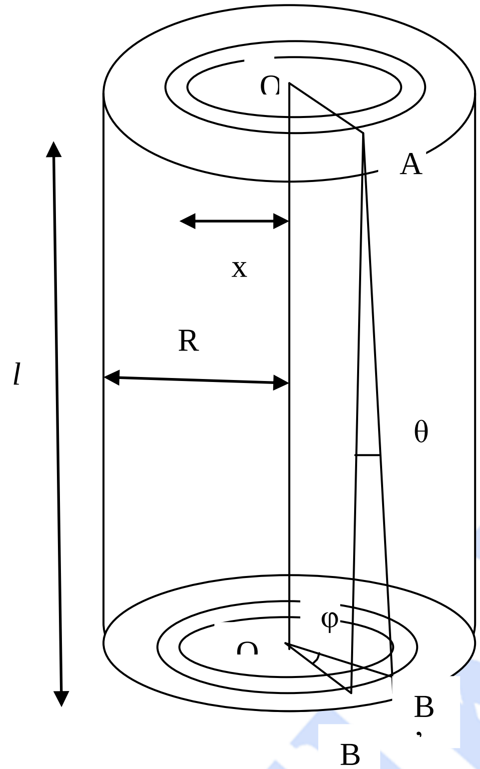
$$\tau = \frac{\pi\eta r^4}{2l}$$

Ans

Consider a cylinder rod of length l and radius r of a material of coefficient of rigidity η. Let the upper end of the rod is fixed rigidly and at the lower free end a twisting couple is applied in an anticlockwise direction in a plane perpendicular to the length of the rod which results each cross section of the rod rotates about the axis of the rod through a same angle called angel of twist.

As rod is twisted, a restoring couple is set up in it. In equilibrium state, this restoring couple is equal and opposite of the twisting couple. To find this couple let the cylindrical rod is divided in to a large no. of thin co-axial cylindrical shells an consider one such shell of radius x and thickness dx.

Before twisting the rod, let AB be a line parallel to the axis of the rod and point B shited to B′ when rod is twisted so that the line AB shifts to AB′. Thus, the twisting couple shear the cylindrical shell trough an angel BAB′= θ. This angel is called angel of shear.



Hebce, in ΔBO′B′, we have

Arc BB′ = angel x radius = φ x O′B = x φ

But in ΔBAB′, we have

Arc BB′ = angel x radius = θ x AB = lθ

Hence, x φ= lθ

Or $\theta = \frac{x\varphi}{l}$

Now let F be the tangential force acting on the base of elementary cylindrical shell. Since face area of this shell is 2πxdx, the shearing stress acting is given by

$$Shearing\ stress = \frac{Force}{Area} = \frac{F}{2\pi x dx}$$

Hence $\eta = \frac{Shearing\ stress}{Shearing\ strain} = \frac{\frac{F}{2\pi x dx}}{\frac{x\varphi}{l}} = \frac{Fl}{2\pi x^2 \varphi dx}$

Or

$$F = \frac{2\pi\varphi\eta}{l} x^2 dx$$

The moment of this force about axis OO' of the rod = force X distance

$= Fx = \frac{2\pi\varphi\eta}{l} x^3 dx$

This is the required couple to twist the elementary shell of radius x through an angel φ. Therefore, the twisting couple , required to twist the whole rod of radius r, is obtained as

$$\tau_s = \int_0^r \frac{2\pi\varphi\eta}{l} x^3 dx = \frac{2\pi\varphi\eta}{l} \left[\frac{x^4}{4}\right]_0^r = \frac{\pi\varphi\eta r^4}{2l}\ radian$$

The couple required is thus proportional l to twist φ.

Hence the couple required to produce one unit twist is

$$C_s = \frac{\pi\eta r^4}{2l}$$

This is called the torsional rigidity or restoring couple per unit radian twist of solid cylinder (or wire).

Q.2 Prove that the hollow shaft is stronger than solid shaft of same material, mass and length.

Ans

As we know that the couple necessary to twist the rods given by

$$\tau = \frac{\pi\eta r^4}{2l}\theta$$

L is the length of the solid cylinder of radius r, θ is the angel of twist and **η** is the modulus of rigidity of the rod.

Hence the twisting couple per unit twist of the cylinder or wire called the torisional rigidity of the material is given by

$$C_{Solid} = \frac{\tau}{\theta} = \frac{\pi\eta r^4}{2l}$$

If a hollow cylinder of the same material, same mass and same length is taken with inner and outer radius r_1 and r_2 respectively, then

$$C_{Hollow} = \frac{\pi\eta({r_2}^4 - {r_1}^4)}{2l}$$

Hence

$$\frac{C_{Hollow}}{C_{Solid}} = \frac{({r_2}^4 - {r_1}^4)}{r^4} = \frac{({r_2}^2 - {r_1}^2)({r_2}^2 + {r_1}^2)}{r^4}$$

Since masses of the both the cylinders are same, and they are made of same material, then we have

$${r_2}^2 - {r_1}^2 = r^2$$

Hence

$$\frac{C_{Hollow}}{C_{Solid}} = \frac{r^2\,({r_2}^2 + {r_1}^2)}{r^4} = \frac{r^2 + 2{r_1}^2}{r^2} = 1 + \frac{2{r_1}^2}{r^2}$$

$$\frac{C_{Hollow}}{C_{Solid}} > 1$$

Or

$$C_{Hollow} > C_{Solid}$$

i.e. the twisting couple per unit twist of the hollow cylinder is greater than that for solid cylinder of the same mass, material and length. hollow shaft is stronger than solid shaft of same material, mass and length.

Q.3 Define Young's modulus, Bulk modulus K and Poisson's ratio σ. Prove that:- Y= 3K(1-2σ).

Ans **Young's modulus:** When the deforming force is applied to a body along one direction only, the strain produced in that direction is called longitudinal strain and corresponding stress is called longitudinal stress. The ratio of longitudinal stress to the corresponding longitudinal strain within the limits of Hook's law is called the Young's modulus of the material of the body. Thus, if a force is applied normally to the cross sectional area A of the wire of length L and the length of wire be increased by l, then

$$\text{longitudinal stress} = \frac{\text{Force applied}}{\text{Cross sectional area}} = \frac{F}{A}$$

$$\text{longitudinal strain} = \frac{\text{Change in length}}{\text{Original Length}} = \frac{l}{L}$$

$$\text{Young's Modulus} = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{l/L} = \frac{FL}{lA}$$

If l=L=1 unit, and A= 1 unit, then Y=F

i.e. Young's modulus for a material is the force required to increase unit length of wire, of unit area of cross section, by unity.

Bulk modulus: When a body is subjected to a uniform pressure perpendicular to its whole surface, it undergoes a change in volume, its shape remains unchanged. The pressure applied gives the normal stress and the change in volume per unit volume gives volume strain. The ratio of normal stress to the corresponding volume strain within the limits of Hook's law is called the Bulk modulus of the material of the body and is denoted by K.

If a force F is applied uniformly and normally on a total surface area A of the body, causing a change of volume v in its original volume V, then

$$\text{normal stress} = \frac{\text{Force applied}}{\text{Cross sectional area}} = \frac{F}{A} = P = \text{Pressure applied}$$

$$\text{Volume strain} = \frac{\text{Change in Volume}}{\text{Original Volume}} = \frac{v}{V}$$

$$\text{Bulk Modulus} = -\frac{\text{Normal stress}}{\text{Volume strain}} = \frac{F/A}{v/V} = -\frac{P}{v/V} = -\frac{PV}{v}$$

Poisson's ratio: When a force acts upon a body along any direction, the size of the body changes not only in that direction but in other direction also. If a force produces elongation in its own direction, a contraction also occurs in a direction perpendicular to it or vice-versa. The change in dimension per unit dimension along which the force acts is called the longitudinal strain and the change in lateral dimension per unit lateral dimension is called lateral strain. "Within elastic limits, the ratio of the lateral strain to the longitudinal strain is a constant for the material of the body and is known as Poisson's ratio and is denoted by σ."

$$\text{Poission's Ratio } (\sigma) = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

It is a dimensionless quantity and has no unit.

Relation between Y, K and σ:

Consider a cube of unit side and let a force P acts normally outward on each of its six faces. Each force produces extension in its own direction and compression in perpendicular directions.

Since

$$\text{Young's Modulus, } Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{P}{\text{longitudinal strain}}$$

So,

$$\text{longitudinal strain} = \frac{P}{Y}$$

and

$$\text{Poission's Ratio } (\sigma) = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$\text{Poission's Ratio } (\sigma) = \frac{\text{lateral strain}}{\frac{P}{Y}}$$

$$\text{Lateral Strain} = \frac{\sigma P}{Y}$$

Now if we consider force parallel to the x-axis, then extension will be in x direction and compression along y and z direction.

i.e. Extension along the x-axis = $\frac{P}{Y}$

and Compression along y and z-axes = $-\frac{\sigma P}{Y}$

Similarly, the pairs of forces acting along Y and Z-axes, we have total extensions as:

$$\text{Extension along x} - \text{axis} = \frac{P}{Y} - \frac{\sigma P}{Y} - \frac{\sigma P}{Y} = \frac{P}{Y}(1 - 2\sigma)$$

$$\text{Extension along y} - \text{axis} = -\frac{\sigma P}{Y} + \frac{P}{Y} - \frac{\sigma P}{Y} = \frac{P}{Y}(1 - 2\sigma)$$

$$\text{Extension along z} - \text{axis} = -\frac{\sigma P}{Y} - \frac{\sigma P}{Y} + \frac{P}{Y} = \frac{P}{Y}(1 - 2\sigma)$$

Thus each side of the cube becomes$1 + \frac{P}{Y}(1 - 2\sigma)$.

Thus the change in volume = $\left\{1 + \frac{P}{Y}(1 - 2\sigma)\right\}^3 - 1$

$$= 1 + \frac{3P}{Y}(1 - 2\sigma) - 1$$

$$= \frac{3P}{Y}(1 - 2\sigma)$$

Neglecting the higher order of σ^2.

Now

$$\text{Volume strain} = \frac{\text{Change in Volume}}{\text{Original Volume}} = \frac{3P}{Y}(1 - 2\sigma)$$

$$\text{Bulk Modulus K} = \frac{\text{Normal stress}}{\text{Volume strain}} = \frac{P}{\frac{3P}{Y}(1 - 2\sigma)}$$

$$K = \frac{1}{\frac{3}{Y}(1-2\sigma)} = \frac{Y}{3(1-2\sigma)}$$

$$Y = 3K(1-2\sigma)$$

Q.4 Define Young's modulus of elasticity, modulus of rigidity and Poisson's ratio. Prove that

(i) $\boldsymbol{Y = 2\eta(1+\sigma)}$ **(ii)** $\boldsymbol{\sigma = \frac{3K-2\eta}{2\eta+6K}}$

Ans

Modulus of rigidity: When external forces act upon a body in such a way as to change only the shape of the body, its size remaining unchanged, the body is said to be sheared. This takes place by the movement of layers one over the other, under the action of impressed forces which are tangential to the surface of the body. Let us consider a cube in which lower face is fixed and a tangential force is applied to the upper surface along x- direction. The layers parallel to the two faces slide over one another, each layer sliding by an amount proportional to its distance from the lower fixed base and thus the lines joining the two faces turn through certain angel θ. This angel θ is known as shearing strain or shear.

$$Shear\ Strain\ \theta = \frac{Lateral\ Displacement\ of\ the\ upper\ plane}{Its\ distance\ from\ fixed\ surface} = \frac{x}{L}$$

The ratio of shearing stress to the shearing strain, within the limits of Hooke's law, is called modulus of rigidity of the material of the body and is denoted by **η**.

$$\eta = \frac{\text{Shearing Stress}}{\text{Shearing Strain}} = \frac{F/A}{\theta}$$

Consider a cube of unit side and let a force P acts normally outward on faces parallel to x-axis and inward on faces parallel to y-axis. Each force produces extension in its own direction and compression in perpendicular directions. Since

$$\text{Young's Modulus}, Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{P}{\text{longitudinal strain}}$$

So,

$$\text{longitudinal strain} = \frac{\text{P}}{\text{Y}}$$

and

$$\text{Poission's Ratio } (\sigma) = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$\text{Poission's Ratio } (\sigma) = \frac{\text{lateral strain}}{\frac{\text{P}}{\text{Y}}}$$

$$\text{Lateral Strain} = \frac{\sigma\text{P}}{\text{Y}}$$

Now if we consider outward force parallel to the x-axis, then extension will be in x direction and compression along y and z direction.
i.e. Extension along the x-axis = $\frac{\text{P}}{\text{Y}}$
and Compression along y and z-axes = $-\frac{\sigma\text{P}}{\text{Y}}$
Similarly, if we consider outward force parallel to the x-axis, then extension will be in x direction and compression along y and z direction.
Compression along the y-axis = $-\frac{\text{P}}{\text{Y}}$
and Extension along x and z-axes = $\frac{\sigma\text{P}}{\text{Y}}$

$$\text{Extension along x} - \text{axis} = \frac{\text{P}}{\text{Y}} + \frac{\sigma\text{P}}{\text{Y}} = \frac{\text{P}}{\text{Y}}(1+\sigma)$$

$$\text{Extension along y} - \text{axis} = -\frac{\sigma\text{P}}{\text{Y}} - \frac{\text{P}}{\text{Y}} = -\frac{\text{P}}{\text{Y}}(1+2\sigma)$$

$$\text{Extension along z} - \text{axis} = +\frac{\sigma\text{P}}{\text{Y}} - \frac{\sigma\text{P}}{\text{Y}} = 0$$

Thus we have equal extension and compression along X and Y-axes.
But the sum of simultaneous equal compression and extension at right angles to each other are equivalent to a shear θ. Hence

$$\theta = \frac{\text{P}}{\text{Y}}(1+\sigma) + \frac{\text{P}}{\text{Y}}(1+\sigma) = \frac{2\text{P}}{\text{Y}}(1+\sigma)$$

$$\frac{P}{\theta} = \frac{Y}{2(1+\sigma)}$$

Further , the extensional stress P and the compressional stress P at the right angles to each other are equivalent to a shearing stress P.

Therefore, the modulus of rigidity

$$\eta = \frac{\text{Shearing Stress}}{\text{Shearing Strain}} = \frac{P}{\theta} = \frac{Y}{2(1+\sigma)}$$

$$Y = 2\eta\,(1+\sigma)$$

Since we also have

$$\mathrm{Y} = 3\mathrm{K}(1-2\sigma)$$

$$\frac{\mathrm{Y}}{3\mathrm{K}} = (1-2\sigma)$$

And

$$\frac{Y}{\eta} = 2\,(1+\sigma) = 2+2\sigma$$

On adding the above two equations, we get

$$\frac{\mathrm{Y}}{3\mathrm{K}} + \frac{Y}{\eta} = 3$$

Or

$$Y = \frac{9\eta\mathrm{K}}{3K+\eta}$$

Also we have

$\mathrm{Y} = 3\mathrm{K}(1-2\sigma)$ and $Y = 2\eta\,(1+\sigma)$

From above these equation, on eliminating Y,

$$3\mathrm{K}(1-2\sigma) = 2\eta\,(1+\sigma)$$

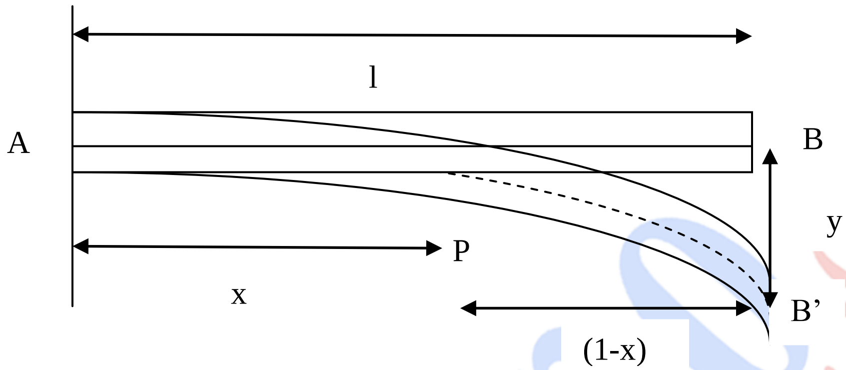
$$\sigma = \frac{3K-2\eta}{6K+2\eta}$$

Q.3 What is cantilever? A weight W is suspended at one ends of the cantilever of length L. Find the formula for depression δ at the end.

Ans

A beam fixed horizontally at one end and loaded at the other is called a cantilever.

Let AB represent the neutral axis of a cantilever of length L, fixed at the end A and loaded at B with mass M. The end B is thus depressed downward compared to A and the neutral axis takes up the new position AB′. Let the weight of the beam is negligible and produces no bending.



Let us take X-axis in the direction of AB and Y- axis vertically downward with the fixed end A as origin. Let coordinate of point P (x,y). The distance of point P from free end is (L-x) and hence the moment of external couple at this section due to load Mg is = Mg (L-x)
For equilibrium of this section at point P, the moment of external couple must be equal to the internal bending moment YI/R where R is the radius of curvature of the neutral axis at the section. Hence

$$\frac{YI}{R} = Mg\,(L - x)$$

Or

$$\frac{1}{R} = \frac{Mg\,(L - x)}{YI}$$

If y be the depression of the beam at the section at P, the radius of curvature at this section is given by

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}$$

Where $\frac{dy}{dx}$ is the slope of the tangent at point (x,y). Since the depression is taken very small, so $\left(\frac{dy}{dx}\right)^2$ is negligible compared to unity. Thus

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

From above equations, we have

$$\frac{d^2y}{dx^2} = \frac{Mg\,(L-x)}{YI}$$

On integrating this equation we get

$$\frac{dy}{dx} = \frac{Mg\,(Lx - \frac{x^2}{2})}{YI} + D$$

Where D is an arbitrary constant and can be obtained by the initial condition that at x =0, $\frac{dy}{dx} = 0$

Which gives D = 0

Hence

$$\frac{dy}{dx} = \frac{Mg\,(Lx - \frac{x^2}{2})}{YI}$$

Integrating again, we get

$$y = \frac{Mg\,(L\frac{x^2}{2} - \frac{x^3}{6})}{YI} + E$$

Where E is an arbitrary constant and can be obtained by the initial condition that at x =0, y = 0

So, $y = \frac{Mg}{YI}\left((L\frac{x^2}{2} - \frac{x^3}{6})\right)$

If W is the weight suspended at one ends of the cantilever, then W= Mg

Hence

$$y = \frac{W}{YI}\left((L\frac{x^2}{2} - \frac{x^3}{6})\right)$$

Now to calculate the depression at the end, we have at x= L, y = δ

So

$$\delta = \frac{W}{YI}\left((L\frac{L^2}{2} - \frac{L^3}{6})\right)$$

$$\delta = \frac{W}{YI}\frac{L^3}{3}$$

$$\delta = \frac{WL^3}{3YI}$$

Q.4 Two beams of equal length, same material and same mass having square and circular cross section are given. Calculate the ratio of weight required to produce same depression.

Ans The depression formula for a beam when weight W is placed at one end of the beam is given by

$$\delta = \frac{WL^3}{3YI}$$

If two beams of equal length, same material and same mass having square and circular cross section are taken, then their depression depends on moment of Inertia and the weight suspended at the end.

For square cross section, let the breadth and thickness of the beam is a, then moment of Inertia

$$I_{square} = \frac{bd^3}{12} = \frac{a^4}{12}$$

So

$$\delta_{\text{square}} = \frac{W_{square}L^3}{3YI} = \frac{W_{square}L^3}{3Y\frac{a^4}{12}} = 4\frac{W_{square}L^3}{a^4Y}$$

For the beam of circular cross section, let r be the radius of circular cross section, then

$$I_{circular} = \frac{\pi r^4}{4}$$

$$\delta_{\text{circular}} = \frac{W_{curcular}L^3}{3YI} = \frac{W_{curcular}L^3}{3Y\frac{\pi r^4}{4}} = \frac{4W_{curcular}L^3}{3Y\pi r^4}$$

Now if $\delta_{\text{square}} = \delta_{\text{circular}}$

Then

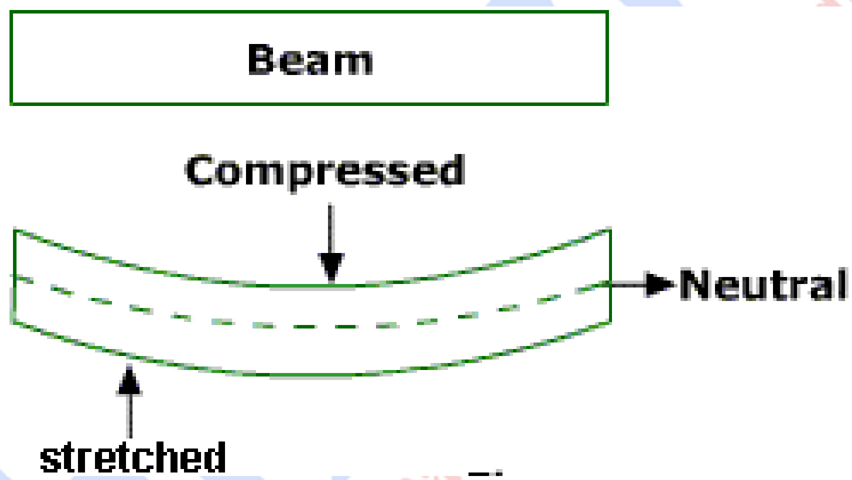
$$4\frac{W_{square}L^3}{a^4Y} = \frac{4W_{curcular}L^3}{3Y\pi r^4}$$

$$\frac{W_{square}}{a^4} = \frac{4W_{curcular}}{3\pi r^4}$$

$$\frac{W_{square}}{W_{curcular}} = \frac{4a^4}{3\pi r^4}$$

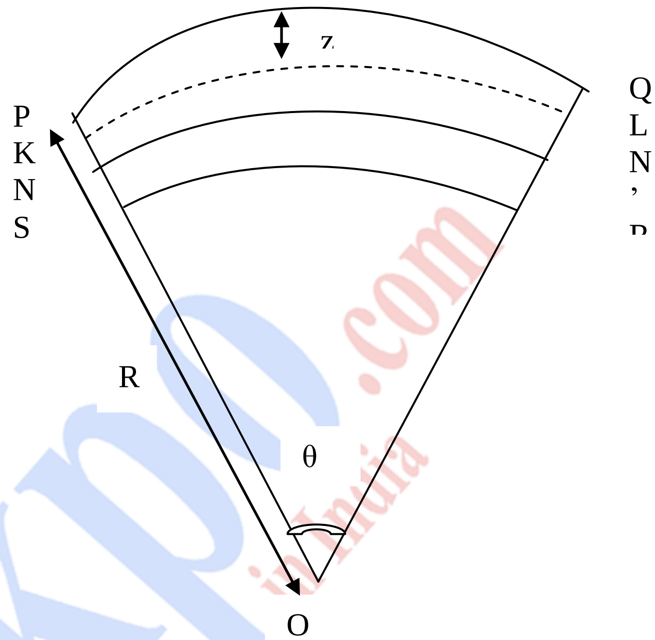
Q.5 Define neutral surface, plane of bending and bending moment and derive an expression for bending moments of a rectangular beam.

Ans **Neutral Surface:** When equal and opposite couples are applied at the ends of a beam in plane parallel to its length, the beam bends into a circular arc. Due to bending the filament of the convex sides are lengthened while those on the concave sides are shortened. There is a layer in between in which the filaments are neither lengthened nor shortened but remain constant in length. This layer is called neutral layer or neutral surface.



Plane of bending: The plane in which the beam bends is called the plane of bending. It is the same as the applied couple. If the beam is placed horizontally, the plane of bending is vertical plane.
Bending moment: Bending moment may be defined as "The total moment of all the forces arising in a bent beam and trying to resist its deformation caused by an external couple."

Expression for bending moment:



Let us consider a small part of the beam bounded by two transverse sections PS and QR. Let R be the radius of curvature of the neutral axis NN' and let it subtend an angel θ at its center of curvature. Consider a filament KL at a distance z from the neutral axis so that KQ = R+z

Thus KL = (R+z) θ

And NN' = R θ

Before bending, the length of the filament KL = NN' = R θ

Hence the extension in the filament KL is = (R+z) θ – R θ = z θ

So, the longitudinal strain of this filament is = $\frac{Increase\ in\ length}{original\ length} = \frac{z\theta}{R\theta} = \frac{z}{R}$

Thus the strain in the filament is directly proportional to its distance from neutral axis.

If Y is the Young's modulus of elasticity of beam, then

$$Y = \frac{Longitudinal\ Stress}{Longitudinal\ Strain}$$

Thus,

Longitudinal Stress = Y X Longitudinal Strain

$$\text{Longitudinal Stress} = \text{Y x } \frac{\text{z}}{\text{R}}$$

Now if 'a' be the area of cross section of the filament, then force acting in this area = stress x area

$$\text{Force} = \text{Y } \frac{\text{z}}{\text{R}}\text{a}$$

The moment of this force about a line through the neutral axis and perpendicular to the plane of bending is given by

$$moment = \text{Y } \frac{\text{z}}{\text{R}}\text{a x z}$$

$$moment = \frac{\text{Ya}}{\text{R}} \text{z}^2$$

The sum of moments of all the forces of extension and compression acting over the whole cross section of the beam i.e. internal bending moment is given by

$$G = \sum \frac{\text{Ya}}{\text{R}} \text{z}^2 = \frac{\text{Y}}{\text{R}} \sum \text{a z}^2 = \frac{\text{YI}}{\text{R}}$$

Where $I = \sum \text{a z}^2$ is a quantity analogous to moment of inertia about the axis z=0. This quantity is known as geometrical moment of inertia of the cross section about an axis through its centroid and perpendicular to the plane of bending.

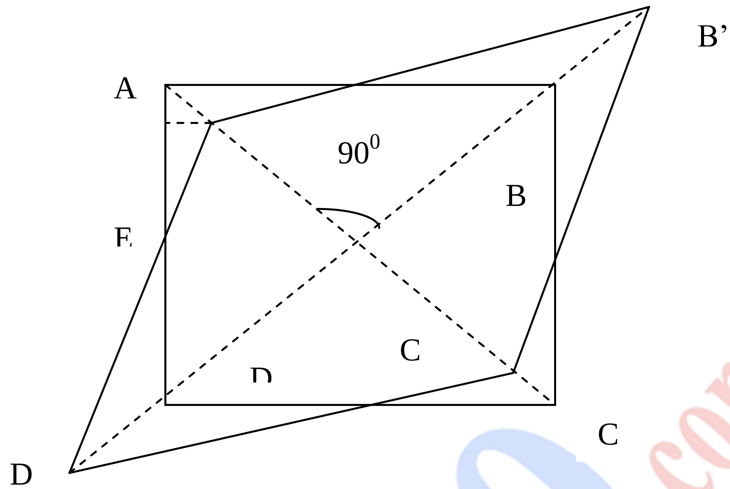
For a rectangular cross sectional beam with breadth 'b' and thickness 'd', the area of cross section is bd and $Radius\ of\ gyration = \frac{d^2}{12}$

Hence $I = bd\ x \frac{d^2}{12} = \frac{bd^3}{12}$

So, the bending moment for rectangular cross sectional beam:

$$G = \frac{\text{Y}}{\text{R}} \frac{bd^3}{12}$$

Q.6 Prove that equivalence of shear to compression and an equal extension strains at right angles to each other.



Let ABCD be the front face of the cube of side L and let it be compressed along the diagonal AC and extended by an equal magnitude along the diagonal BD. Let A′C′ and B′D′ be the new positions of diagonal such that AA′ = CC′ =BB′= DD′= x (say)

Now, OA = AB cos BAO = AB cos45^0 = $\frac{L}{\sqrt{2}}$

Hence, OA′ = OA – AA′ = $\frac{L}{\sqrt{2}} - x$

Similarly, OB′ = OB + BB′ = $\frac{L}{\sqrt{2}} + x$

Since we have

$$(A'B')^2 = (OA')^2 + (OB')^2$$

$$(A'B')^2 = (\frac{L}{\sqrt{2}} - x)^2 + (\frac{L}{\sqrt{2}} + x)^2$$

$$(A'B')^2 = L^2 + 2x^2$$

In general $2x^2$ is extremely small in comparison with L^2 and can be neglected. Therefore

$$(A'B')^2 = L^2$$

$$A'B' = L = AB$$

Thus the length of the sides of the cube remains unchanged under deformation. If now deformed shape A′B′C′D′ is rotated clockwise through an angel AEA′ so that D′C′ coincide with DC, then A′D′ will make an angel $\theta = 2\ AEA'$ with AD. This position represents the case of shear where the plane DC is fixed and the planes above suffer parallel displacements.

Thus the angel of shear $\theta = 2\ AEA' = 2\ \frac{FA'}{EF}$

Now if A′F be the perpendicular from A′ on AD, then from ΔAA′F, we have

$$FA' = AA' sin 45° = \frac{x}{\sqrt{2}}$$

Also AA′ being very small, F will be very near to A and EF may be taken approximately equal to AE. Hence

$$EF = AE = \frac{L}{2}$$

Hence

$$\theta = 2\ \frac{FA'}{EF} = 2\ \frac{\frac{x}{\sqrt{2}}}{\frac{L}{2}} = \frac{2x\sqrt{2}}{L}$$

Now compressional strain along the diagonal AD

$$= \frac{AA'}{AO} = \frac{x}{\frac{L}{\sqrt{2}}} = \frac{x\sqrt{2}}{L} = \frac{\theta}{2}$$

Similarly, extensional strain along diagonal BD

$$= \frac{BB'}{OB} = \frac{x}{\frac{L}{\sqrt{2}}} = \frac{x\sqrt{2}}{L} = \frac{\theta}{2}$$

Thus the compressional and extensional strain is half the angel of shear or the angel of shear is twice the compressional and extensional strain.

Hence, "The sum of simultaneous and equal compressional and extensional strains at right angels to each other is equivalent to a shear, and that the compressional strain or the extensional strain is half the shear strain."

Multiple Choice Questions

1. If the work done in stretching a wire by 1 mm is 2 J, the work necessary for stretching another wire of the same material but with double the radius of cross-section and half the length by 1 mm is in joules

(a) 16
(b) 8
(c) 4
(d) $\frac{1}{4}$

Answer: (a)

2. To compress a liquid by 10% of its original volume, the pressure required is 2×10^5 N/m^2. the bulk modulus of the liquid is

(a) 2×10^4 N/m^2
(b) 2×10^5 N/m^2
(c) 2×10^7 N/m^2
(d) 2×10^6 N/m^2

Answer: (d)

3. The modulus of elasticity is dimensionally equivalent to

(a) Strain
(b) Stress
(c) Surface tension
(d) Poisson's ratio

Answer: (b)

4. If by applying a force, the shape of a body is changed, then the corresponding stress is known as

(a) Tensile stress
(b) Bulk stress
(c) Shearing stress
(d) Compressive stress

Answer: (c)

5. When the tension in a metal wire is T_1, its length is L_1 and when the tension is T_2, its length is L_2. its unscratched length is

(a) $\frac{L_1 + L_2}{2}$

(b) $\frac{T_2L_1 \quad T_1L_2}{T_2 \quad T_1}$

(c) $\sqrt{\frac{L_1L_2}{2}}$

(d) $\frac{L_1T_2 + L_2T_1}{T_1 + T_2}$

Answer: (b)

6. According to Hooke's law of elasticity, within elastic limits, if the stress is increased, the ratio of stress to strain

(a) Increases
(b) Decreases
(c) Becomes zero
(d) Remains constant

Answer: (d)

7. Two wires have the same material and length, but their masses are in the ration of 4:3. If they are stretched by the same force, their elongations will be in the ratio of

(a) 2:3
(b) 3:4
(c) 4:3
(d) 9:16

Answer: (b)

9. The energy stored per unit volume of a strained wire is

(a) $\frac{1}{2} \times (\text{load}) \times (\text{extension})$

(b) $\frac{1}{2} \frac{y}{(\text{strain})^2}$

(c) $\frac{1}{2} y (\text{strain})^2$

(d) stress strain

Answer: (c)

10. In an experiment to determine the Young's modulus of the material of a wire, the length of the wire and the suspended mass are doubled. Then the Young's modulus of the wire

(a) Becomes double
(b) Becomes four time
(c) Remain unchanged

(d) Becomes half

Answer: (c)

11. Which one of he following does not affect the elasticity of a substance?

(a) Hammering

(b) Adding impurity in the substance

(c) Changing the dimensions

(d) Change of temperature

Answer: (c)

12. The bulk modulus of a fluid is inversely proportional to the

(a) Change in pressure

(b) Volume of the fluid

(c) Density of the fluid

(d) Change in its volume

Answer: (d)

13. Shearing strain is given by

(a) Deforming force

(b) Shape of shear

(c) Angle of shear

(d) Change in volume of the body

Answer: (c)

14. Which of the following is dimensionless quantity?

(a) Stress

(b) Young's modulus

(c) Strain

(d) Pressure

Answer: (c)

16. The dimensional formula for modulus of rigidity is

(a) $[M^1 L^1 T^2]$

(b) $[M^1 L^1 T^2]$

(c) $[M^1 L^1 T^2]$

(d) $[M^1 L^2 T^2]$

Answer: (b)

18. For Hooke's law to hold good, the intermolecular distance must be ________ as compared to the equilibrium distance

(a) Much more
(b) Zero
(c) Much less
(d) Approximately same
Answer: (d)

19. The substance which shows practically no elastic effect is
(a) Quartz
(b) Copper
(c) Silk
(d) Rubber
Answer: (a)

Key terms

Motion: change of position with time

Events are physical phenomena that occur independent of any reference frame. For example: a flash, explosion, return of a spaceship, or disintegration of a subatomic particle.

Observers record events, both the time and spatial coordinates, in a particular reference frame. For example, Mission Control in Houston marking down the time and location of the splashdown of a space capsule. The reference frame in this case is the Earth.

Simultaneous events occur when the light signals from two events reach an observer at the same time

Relativity of Simultaneity:

Two events simultaneous in one inertial frame are not simultaneous in any other frame. This is a consequence of Einstein's Postulates.

Proper time is the time difference between two events occurring at the same position (Denoted by t_0 or τ).

Rest frame is the inertial frame where two events are only separated by time. The frame in which the proper time is measured

Proper length is the distance between two positions at rest, the length measured in the rest frame. (Denoted by L_0).

Einstein's Postulates

1. The laws of physics, including electromagnetism, are the same in all inertial frames.
2. Every observer measures the same value *c* for the speed of light (in vacuum) in all inertial frames.

Multistage rocket: A **multistage** (or **multi-stage**) **rocket** is a rocket that uses two or more *stages*, each of which contains its own engines and propellant. A *tandem* or *serial* stage is mounted on top of another stage; a *parallel* stage is attached alongside another stage. The result is effectively two or more rockets stacked on top of or attached next to each other. Taken together these are sometimes called a launch vehicle.

Note: The force acting on the system is conservative if the mechanical energy of the system is conserved.

Principle of rocket : A rocket in its simplest form is a chamber enclosing a gas (fuel and oxidizing agent) under pressure. A small opening at one end of the chamber allows the gas to escape, and in doing so provides a thrust that propels the rocket in the opposite direction. The oxidizing agent of oxygen is carried in the liquid form of oxidizers like H_2O_2 and HNO_3. When the fuel burns, a jet of hot gases emerges forcefully from the tail of the rocket which may be considered as action force. A force is exerted by the jet of hot gases on the rocket which is reaction force. The jet of hot gases acquires momentum in the backward direction while rocket due to reaction force acquires momentum in the forward direction, consequently the rocket gains forward motion. The total momentum of the system is conserved as there is no external force.

Gravitational field: Every particle of matter exerts a force of attraction on every other particle. This force is called gravitational force of attraction and is given by Newton's law of gravitation. The area round about the attracting particle within which its gravitational force of attraction is perceptible is called its gravitational field.

Intensity of gravitational field: The intensity of gravitational field or the gravitational attraction at a point in a gravitational field is the force experienced by unit mass placed at that point in the field. Thus the gravitational field at a point due to a particle of mass M at a distance r from the point is, given by

$$\boldsymbol{f_g} = -\frac{GM}{r^2}\,\hat{\boldsymbol{r}}$$

Where $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$ is a unit vector along **r**.

Thus force per unit mass is a measure of the field intensity.

Gravitational potential energy:

The gravitational potential energy of a system of particles is generally defined as the amount of work that must be done by an external agent to assemble the system starting from an infinite distance to the present position against the gravitational field. The potential energy of the particles is assumed to be zero when they are infinitely apart.

Consider a system of two masses m_1 and m_2. Let them initially be infinitely far from each other. Let us assume that mass m_1 is already present and we bring m_2 from infinity to at the point where mass m_1 is situated. Then the work required to be done by the gravitational force exerted by m_1 on m_2 is

$$W_{12} = -\int_{\infty}^{r_{12}} \boldsymbol{F_{21}}.d\boldsymbol{r} = -\int_{\infty}^{r_{12}} \left(\frac{Gm_1m_2}{r^2}\,\hat{\boldsymbol{r}}\right).dr = \int_{\infty}^{r_{12}} \frac{Gm_1m_2}{r^2}.dr = \left[-\frac{Gm_1m_2}{r}\right]_{\infty}^{r_{12}} = -\frac{Gm_1m_2}{r_{12}}$$

The gravitational potential energy of two particles of mass m1 and m2 placed at a distance r is $\frac{Gm_1m_2}{r}$

If we want to separate the system into isolated masses again, we would have to supply the same amount of energy. Hence the gravitational potential energy of two particles of mass m1 and m2 placed at a distance r is given by $\frac{Gm_1m_2}{r}$

Gravitational potential: If a body is moved in the gravitational field of another body, a certain amount of work is to be done. If moved in the direction of the field, the work is done by the field itself, and if moved against the field, the work is to be done by some external agent.

"The work done in moving a unit mass from infinity to any point in the gravitational field of a body is called the gravitational potential at that point due to the body."

Gravitational potential at a point in gravitational field may also be defined as the potential energy of unit mass placed at that point. It is denoted by V_g. Its value at a distance r from the body of mass M is given by

$$V_g = -\int_{\infty}^{r} \boldsymbol{f_g}.d\boldsymbol{r} = -\int_{\infty}^{r} \left(\frac{GM}{r^2}\,\hat{\boldsymbol{r}}\right).dr = \int_{\infty}^{r} \frac{GM}{r^2}.dr = \left[-\frac{GM}{r}\right]_{\infty}^{r} = -\frac{GM}{r}$$

It must be noted that potential is a scalar quantity while gravitational field is a vector quantity.

Note: angular momentum of a system of particles with respect to O is the vector sum of the angular momentum of the center of mass with respect to O and angular momentum of the system of particles with respect to center of mass.

$$\overrightarrow{J_o} = \overrightarrow{J_{cmo}} + \overrightarrow{J_{cm}}$$

Note: For conservative force:- F= - grad U = $-\vec{\nabla}$U, and curl $\vec{F} = \vec{\nabla} \times \vec{F} = 0$

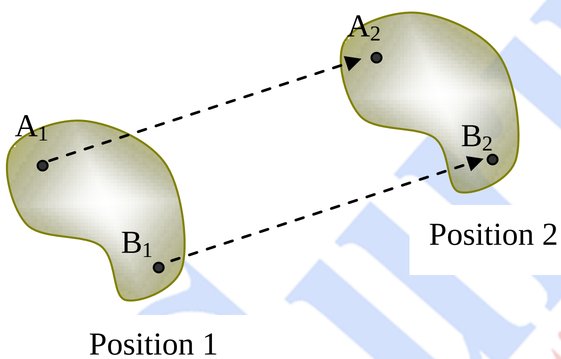
Note: The total kinetic energy of a system of many particles is equal to the sum of kinetic energy of center of mass and total kinetic energy of particles about the center of mass.

Rigid Body Motion:

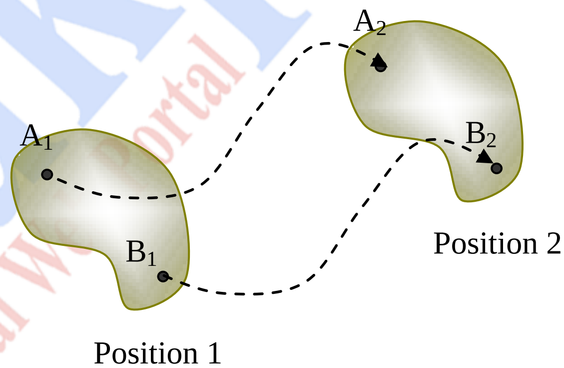
From this point on, the study of dynamics becomes more difficult because instead of looking at a particle body, where you only describe the motion of the center of mass, you will now be working with rigid bodies. Because a rigid body consists of a number of different particles all held tightly together in a fixed position with respect to each other, you will need to be able to describe the velocity and acceleration of the different parts of the body. As a body moves through space and rotates, the velocity and acceleration of the parts of a body will be different.

Translation:

Motion of a body through space, where the body keeps the same relative orientation. In other words, the body moves, but does not rotate.



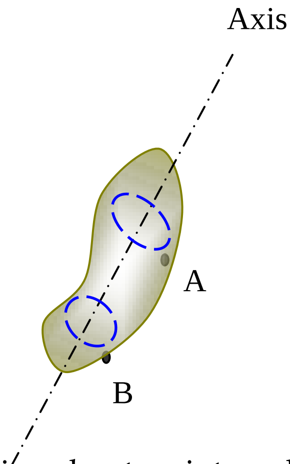
Rectilinear Translation:



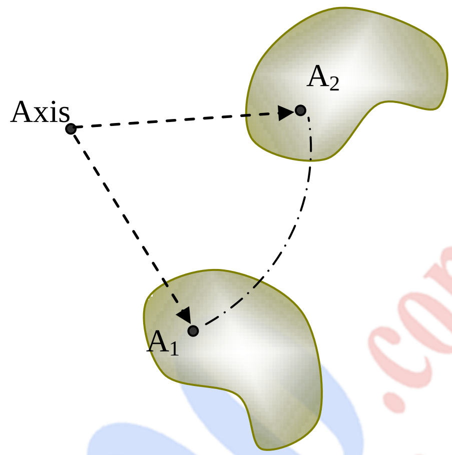
Curvilinear Translation

Rotation:

Motion where each part of a body moves about along circular paths all having the same center of rotation. There is one point in space which shows no motion which occurs along the axis of rotation.

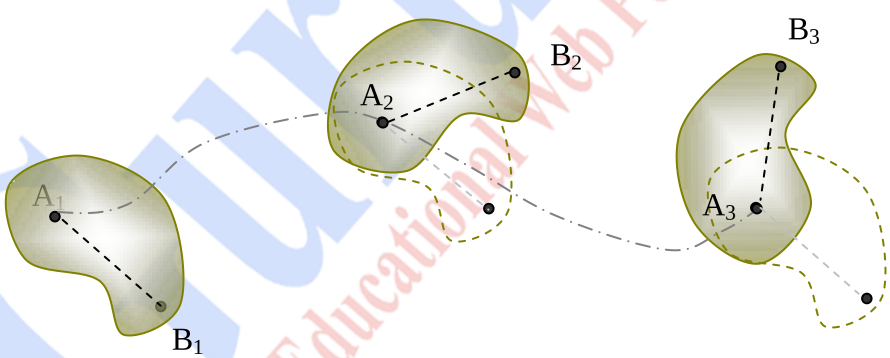


Rotation about an internal axis



Rotation about an external axis

General Plane Motion:
When a body moves such that each point of the rigid body remains in its own plain, but may experience both translation and rotation through the plane.



General Plane Motion: Points A and B both stay plane of the paper.

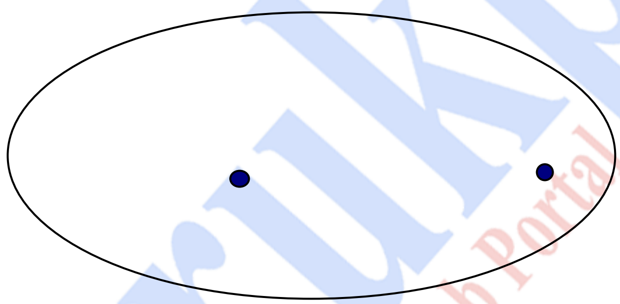
General Motion:
When a rigid body moves such that its parts move with both rotation and translation through 3-dimensional space.

The law of universal gravitation: every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
If the particles have masses *m*1 and *m*2 and are separated by a distance *r*, the magnitude of this gravitational force is

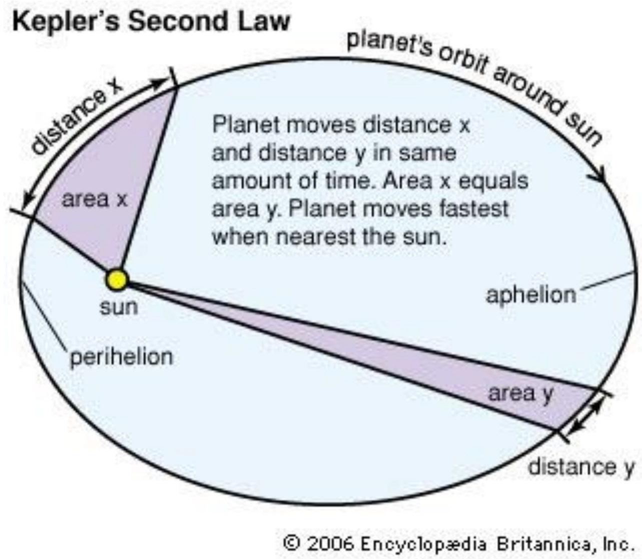
$$F_g = \frac{Gm_1m_2}{r^2}$$

where *G* is a constant, called the *universal gravitational constant,* that has been measured experimentally. Its value in SI units is G= 6.673X $10^{(-11)}$ N-m^2/kg^2

Kepler's law for planetary motion:
Kepler's first law:
Each planet moves in an ellipse with the Sun at one focus.
Ellipse is a closed figure drawn around two points (foci) in such a way that the sum of the distances from any point on the ellipse to the foci equals a constant.



Kepler's second law:
The line between the Sun and the planet sweeps over equal areas in equal time intervals.
Speed of a planet along an orbit is not constant: a planet moves faster when closer to the Sun and slower, when farther away.



Kepler's third law

*The ratio of the cube of the semimajor axis (**a**) to the square of the period of revolution (**P**) is the same (**K**) for each planet.*

$$\frac{a^3}{P^2} = K$$

or

the ratio of the squares of the periods of any two planets revolving about the sun is equal to the ratio of the cubes of their average distances from the sun, where T is period, r is average distance from the sun, and a and b represent two planets; provided accurate data for the distance the planets are from the sun.

$$\left(\frac{T_a}{T_b}\right)^2 = \left(\frac{r_a}{r_b}\right)^3$$

Gravitational mass: Gravitational mass of the body can be defined as the mass of a body in terms of the gravitational force of attraction exerted on it by another body (eg. earth) i.e.

$$F = \frac{GMm_g}{r^2}$$

Here $m_g$is the gravitational mass and M is the mass of the earth and r is the distance of the body from the centre of the earth.

Inertial mass: Newton's first law of motion states that "Every body continues to be in its state of rest or uniform motion until and unless acted upon by the external forces". This inertness of the body to change its state of rest or of uniform motion is called inertia and is an inherent property of the body by virtue of its mass called inertial mass and can be measured as the ratio of the force applied (F) and acceleration (a) produced in the body". i.e.

$$m_i = \frac{F}{a}$$

Here m_i is the inertial mass of the body.

Central forces : The forces which act on a particle along a line joining the centre of mass of the particle to some fixed point are called the central forces and the point is known as centre of force.
Note: When a particle moves under the action of central force, its angular momentum is conserved.

Conservation of angular momentum:
We know that the external torque applied to a system of particles is given by

$$\vec{\tau} = \frac{d\vec{L}}{dl}$$

Now if $\vec{\tau} = 0$, then $\frac{d\vec{L}}{dl} = 0, \vec{L} = constant$

i.e. when the external torque applied to the system is zero, the total angular momentum of the system remains conserved. This is the law of conservation of angular momentum.

Rigid body : A rigid body is an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it.

Moment of inertia: Moment of inertia is the name given to rotational inertia, the rotational analog of mass for linear motion. It appears in the relationships for the dynamics of rotational motion. Moment of inertia is a property of a body that defines its resistance to a change in angular velocity about an axis of rotation. The tendency of a body to resist angular acceleration, expressed as the sum of the products of the mass of each particle in the body and the square of its perpendicular distance from the axis of rotation.

Radius of gyration: the radius of gyration of a body about an axis of rotation is defined as the radial distance of a point from the axis of rotation at which, if the whole mass of the body is assumed to be concentrated, its moment of inertia about the given axis would be the same as with its actual distribution of mass

If m is the mass of the body, its moment of inertia I in terms of its radius of gyration 'K' can be written as :

$$I = mK^2$$

OR

$$K = \sqrt{\frac{I}{m}}$$

Physical significance: the total mass of a rotating body may be supposed to be concentrated at a radial distance 'k' from the axis of rotation so far as the moment of inertia of the body about that axis is concerned

Precession: A symmetrical object rotating about an axis which is fixed at one point is called a top. The axis of rotation of a top can itself rotate about the fixed point. The angular velocity of the axis of rotation of the top, called the precessional angular velocity is assumed to be negligible as compared to the angular velocity of the top about its axis.

Principal axis : A principal axis may be simply defined as one about which no net torque is needed to maintain rotation at a constant angular velocity.

Young's modulus: When the deforming force is applied to a body along one direction only, the strain produced in that direction is called longitudinal strain and corresponding stress is called longitudinal stress. The ratio of longitudinal stress to the corresponding longitudinal strain within the limits of Hook's law is called the Young's modulus of the material of the body.

$$\text{longitudinal stress} = \frac{\text{Force applied}}{\text{Cross sectional area}}$$

$$\text{longitudinal strain} = \frac{\text{Change in length}}{\text{Original Length}}$$

$$\text{Young's Modulus} = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

Bulk modulus: When a body is subjected to a uniform pressure perpendicular to its whole surface, it undergoes a change in volume, its shape remains unchanged. The pressure applied gives the normal stress and the change in volume per unit volume gives volume strain. The ratio of normal stress to the corresponding volume strain within the limits of Hook's law is called the Bulk modulus of the material of the body and is denoted by K.

$$\text{normal stress} = \frac{\text{Force applied}}{\text{Cross sectional area}} = \text{Pressure applied}$$

$$\text{Volume strain} = \frac{\text{Change in Volume}}{\text{Original Volume}}$$

$$\text{Bulk Modulus} = -\frac{\text{Normal stress}}{\text{Volume strain}}$$

Modulus of rigidity: When external forces act upon a body in such a way as to change only the shape of the body, its size remaining unchanged, the body is said to be sheared. This takes place by the movement of layers one over the other, under the action of impressed forces which are tangential to the surface of the body.

$$Shear\ Strain\ \theta = \frac{Lateral\ Displacement\ of\ the\ upper\ plane}{Its\ distance\ from\ fixed\ surface}$$

Poisson's ratio: When a force acts upon a body along any direction, the size of the body changes not only in that direction but in other direction also. If a force produces elongation in its own direction, a contraction also occurs in a direction perpendicular to it or vice-versa. The change in dimension per unit dimension along which the force acts is called the longitudinal strain and the change in lateral dimension per unit lateral dimension is called lateral strain.
"Within elastic limits, the ratio of the lateral strain to the longitudinal strain is a constant for the material of the body and is known as Poisson's ratio and is denoted by σ."

$$\text{Poission's Ratio } (\sigma) = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

Neutral Surface: When equal and opposite couples are applied at the ends of a beam in plane parallel to its length, the beam bends into a circular arc. Due to bending the filament of the convex sides are lengthened while those on the concave sides are shortened. There is a layer in between in which the filaments are neither lengthened nor shortened but remain constant in length. This layer is called neutral layer or neutral surface.

Plane of bending: The plane in which the beam bends is called the plane of bending. It is the same as the applied couple. If the beam is placed horizontally, the plane of bending is vertical plane.

Bending moment: Bending moment may be defined as "The total moment of all the forces arising in a bent beam and trying to resist its deformation caused by an external couple."

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