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Concept based notes

Optimization

(B.Sc)

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this Endeavour. They played an active role in coordinating the various stages of this Endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Linear Programming Problems

Linear Programming – The word 'linear' implies 'proportional' i.e. representation on a line and programming indicates to the process of determining a particular plan of action. This linear programming is a technique which is applicable to those programming problems in which the desired objective as well as the restrictions on the resources give rise to linear functions. Such type of problem is called a linear programming problem.

A business or an industry concern has to function under various limitations of its available resources. To earn maximum profit working under the available resource linear programming is a technique to find the conditions which would maximize or optimize profit.

The general problem of linear programming was first developed and applied along with the simplex method in 1947 George B. Dantzig, Marshall Wood and their associates of the U.S. Department of the air force.

Description of Linear programming problem:

The mathematical description of a linear programming problem (LPP) includes :

(1) A set of simultaneous linear Equations or inequalities which represent the conditions of the problem and

(2) A linear function which expresses the objective of the problem.

A linear programming problem in n variables, with some given restrictions can be stated as follows :

Find $x_1, x_2, \ldots\ldots\ldots, x_n$ which

Optimize $Z = C_1 x_1 + C_2 x_2 + \ldots\ldots + C_n x_n$

Subject to the conditions

$a_{11} x_1 + a_{12} x_2 + ...+ a_1n\, xn \leq$ (or = or $\geq$) b_1

$a_{21} x_1 + a_{22} x_2 + ...+ a_2n\, xn \leq$ (or = or $\geq$) b_2

....

$am_1 x_1 + am_2 x_2 + ... + amn\, x_n \leq$ (or= or $\geq$) bm

and non-negative restrictions

$xj \geq O$, j= 1,2 ......n.

where aij 's and Cj's are constant and x j's are variables

__Some Definations related with LPP –__

Objective function : In a LPP the linear function of the variables which is to optimized is called its objective function.

Constraints: In a LPP the set of a given conditions which are in the form of linear Equations or inequalities, are called its constraints or restrictions.

Non- Negative Restriction :

In a L.P.P. all variables are non negative.

Feasible Solution (F.S.)

A set of value of the variables of a L.P.P. which satisfies the set of constraints and the non- negative restrictions is called a feasible solution.

Optimal solution :

A feasible solution of a L.P.P. which optimizes its objective function is called an optimal solution of the problem.

Basic Solution (B.S.)

Definition – Let AX=b be a system of a m equations in n variables (m< n) and r (A) = r [A:b] = m Then a solution obtained by setting any (n-m) variables to zero is called a basic solution, provided the determinant of the coefficients of the remaining m variables, is not zero.

Thus if B is the matrix of the selected vectors and XB is the vector of the corresponding variables, then the solution of the resulting system is given by –

$BX_B = b$ or $X_B = B^{-1} b$

Thus a solution in which the vectors associated to m variables (basic) are L.I. and the remaining (n-m) variables (non basic) are zero, is called a basic solution

The number of the Basic solution will be at the most

$n_{cm} = \frac{n!}{m!(n-m)!}$

Basic Solutions are of two types :

Non degenerate – A basic solution is called non-degenerate if none of the basic variables vanishes. In other words all the m basic variables must be non-zero and n-m zero variables.

Degenerate – A basic solution is called degenerate if one or more of the basic variables are zero.

Basic Feasible solution (B.F.S.)

Definition : In a linear programming problem, a feasible solution is called a basic feasible solution if it is also basic.

B.F.S are of two types :

Degenerate B.F.S. – A B.F.S. is said to be degenerate B.F.S. if at least one of the basic variables is zero.

Non Degenerate B.F.S. – A B.F.S. is called non-degenerate B.F.S. if non of the basic variable vanishes.

Convex Set

Convex combination

The convex combination of a finite number of points x_1, x_2 ...., xn is defined as a point

$X = \lambda_1 x_1 + \lambda_2 x_2 + ...+ \lambda n\ xn$

Where λi is real and $\geq O\ \forall i$ and $\lambda i = \perp$

the convex combination $i = 1$

of two points x_1 and x_2 is given by

$X = \lambda_1 x_1 + \lambda_2 x_2 ; \lambda_2 x_2 ; \lambda_1, \lambda_2 \geq O$ and $\lambda_1 + \lambda_2 = \perp$

It can also be written as

$X = \lambda X_1 + (1-\lambda) X_2; O \leq \lambda \leq \perp$

This show that the segment of the two points X_1 and X_2 is the set of all possible convex combinations of the two points X_1 and X_2

Convex Polyhedron :

Definition : The set of all convex combinations of a finite number of points is called the convex polyhedron spanned by these points.

Example – The set of all the points of the area of a triangle is a convex polyhedron generated by its vertices. The vertices are also the extreme points.

Simplex : Definition :

An n- dimensional convex polyhedron having exactly (n+1) vertices is called a Simplex.

Example – A point is a simplex in zero dimension . A line is a simplex in one dimension.

Slace and surplus variables :

We know that the constraints of a linear programming problem (L.P.P.) may involve any of the three sign $\leq$, =, $\geq$. In case the constraints are inequalities they can be changed to equations by adding or subtracting the left hand side of each such constraint by non-negative variables. These new variables, if they are Added, are called slace variables and in case when they are subtracted, they are called Surplus variables.

Example - If we have the constraints

$2x_1 - 3x_2 \leq 7$

$5x_1 + x_2 \geq 5$

Now we can changed them into the form of equations by introducing two new variables X_3 and X_4 as follows

$2x_1 - 3x_2 + x_3 = 7$

$5x_1 + x_2 - x_4 = 5$

Here x_3 is slace variable whereas x_4 is a surplus variable.

Q.1 Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and young ones lay 5 eggs per week, each egg being worth 30 paisa. A hen costs Rs. 1 per week to be fed there are only 80 available to spend on purchasing the hens and it is not possible to house more than 20 hens at a time. Formulate the lineur programming problem and solve it by the graphical method to find how many of each kind of hens should be bought in order to have a maximum profit per week.

Sol. – The given Data can be written as

Hens	Cost of each hens	Production of egg	Fed Expenditure	Max. quantity of expenditure	Max. space available
Old	2 Rs.	3	1 Rs.	80 Rs.	20
Young	5 Rs.	5	1 Rs.		
cost of each egg	30 paisa				

Let the person buy x_1 young hens and x_2 old hens.

Then total cost $2x_1 + 5x_2$

Since only 80 Rs. available to spend on purchasing the hens therefore constrained

$2x_1 + 5x_2 \leq 80$

again house available 20 for hens at a time. therefore const. $x_1 + x_2 \leq 20$ and $x_1 \geq 0$, $x_2 \geq 0$

The no. of eggs given by old hens in a week = $3x_1$

and the no. of eggs given by young hens in a week= $5x_2$

The total income by selling the eggs in a week = $(3x_1 + 5x_2)\ \frac{.30}{100}$ Rs.

The total Expenditure for one week = $(x_1 + x_2)$ Rs.

Total weekly profit Z = $\frac{30}{100} \times 3x_1 + \frac{.30}{100} \times 5x_1 - (x_1 + x_2)$

$Z = -.1x_1 + .5x_2$

Hence the mathematical formulation of the given L.P.P. is as follows

Max. $Z = -.1\ x_1 + .5\ x_2$ (objective function)

subject to $2x_1 + 5x_2 \leq 80$ (constraints

$x_1 + x_2 \leq 20$

$x_1, x_2 \leq 20$

and $x_1, x_2 \geq 0$

graphical solution

$2\ x_1 + 5\ x_2 = 80$ or $\frac{x1}{40} + \frac{x2}{16} = 1$ - (1)

$x_1 + x_2 = 20$ or $\frac{x1}{20} + \frac{x2}{20} = 1$ - (2)

$x_1 = 0$ – (3) $x_2 = 0$ – (4)

First we draw these lines in two dimensional space which correspond to the inequalities of the constraints.

Now on considering the solution space for each of given inequalities, we observe that the feasible region, i.e. their common solution space is the shaded area

OABC. Further we find that the co-ordinates on the vertices of the polygon OABC are

0 (0,0); A (20,0) ; B $(\frac{20}{3}, \frac{40}{3})$; C (0,16)

Now the maximum value of Z is at one of the vertices of the polygon OABC. But we have at A (20,0)   Z = -0.1 X20 + 0.5X0 = -2

B (20/3, 40/3)   Z =-0.1 X 20/3+0.5X40/3 = 6

C (0, 16)   Z = 0.1X0+ 0.5X16 =8

0 (0,0)   Z = 0.1X0 +0.5X0 =0

Thus Z is maximum at C (0,16) 8 Rs. it means for obtained maximum profit per week the person should purchase only 16 young hens.

Q. 2 A firm manufactures headache pills in two size A and B size. A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. Size B contain respectivity 1,8 and 6 grain. It has been found by users that it requires at least 12 grains of Asprin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate relief. Formulate the L.P.P and determine graphically the least no. of pills a patient should have to get immediate relief.

Sol. Suppose the patient should take x_1 pills of A type and x_2 pills of B type for immediate relief. therefore

Z= x_1+ x_2

According to question for providing immediate relief patient requires at least 12 grains of asprin, 74 grains of bicarbonate and 24 grains of codeine, therefore

$2x_1 + x_2 \geq 12$

$5x_1 + 8x_2 \geq 74$

$x_1 + 6\,x_2 \geq 20$

and $x_1 \geq 0$, $x_2 \geq 0$,

Hence the mathematical formulation of a given LPP is as follows –

minimize   Z= x_1+ x_2

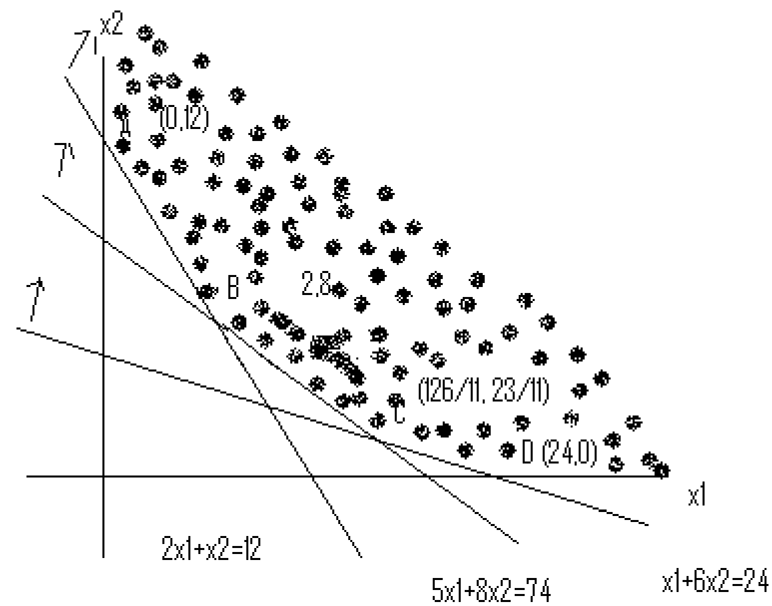
$2x_1 + x_2 \geq 12$

$5x_1 + 8x_2 \geq 74$

$x_1 + 6x_2 \geq 24$

and $x1$, $x2 \geq 0$

graphical solution :-



$2x_1 + x_2 = 12$ or $\frac{x1}{6} + \frac{x2}{12} = 1$ - (1)

$5x_1 + 8x_2 = 74$ or $\frac{x1}{14.8} + \frac{x2}{9.25} = 1$ - (2)

$x_1 + 6x_2 = 24$ or $\frac{x1}{24} + \frac{x2}{4} = 1$ - (3)

$x_1 = 0$ (4) $x_2 = 0$

First we draw these lines in two dimensional space. Now on considering the solution space for each of given inequalities we obtained their common solution space is the unbounded shaded area ABCD as its corner points. From diagram we find that corner points are

A (24,0) ; B $(\frac{126}{12}, \frac{23}{11})$; C (2,8) ; D (0,12)

The value of the objective function at these points are given by

at A (24,0) Z= 24+0 =24

at B $(\frac{126}{12}, \frac{23}{11})$ Z $= \frac{126}{12} + \frac{23}{11} = 13.5$

at C (2,8) Z= 2+8 =10

at D (0,12) Z = 0+12 =12

Therefore the optimal solution of given problem is $x_1=2$, $x_2=8$ and minz =10

It means the patient should take 2 pills of A size and 8 pills of B size for immediate relief.

Q.3 A film manufacturing two types of electric items A and B can make a profit of Rs. 20 per unit of A and Rs. 30 per unit of B. Each unit of A requires 3 motors and 2 transformers and each unit of B requires 2 motors and 4 transformers. the total supply of their per month as restricted to 210 motors and 300 transformers. Type B is an export model requiring a voltage stabilizer which has a supply restricted to 65 units per month

Sol. Mathematical formulation – Let the firm manufacture x_1 electric items of type A and x_2 electric items of type B per month. So total monthly profit is given by Z = 20 x_1 +30 x_2 Since Each unit of A requires 3 motors, each unit of B requires 2 motors and the total supply of motors per month is restricted to 210

So $3x_1+2x_2 \leq 210$

Further each unit of A and B requires 2 and 4 transformers respectively and its supply is restricted to 300, therefore

$2x_1 + 4x_2 \leq 300$

morevoer the voltage stablizer which is required for item B is restricted to 65, So we have

$x_2 \leq 65$

Hence the given problem is to find x_1 and x_2, which minimize

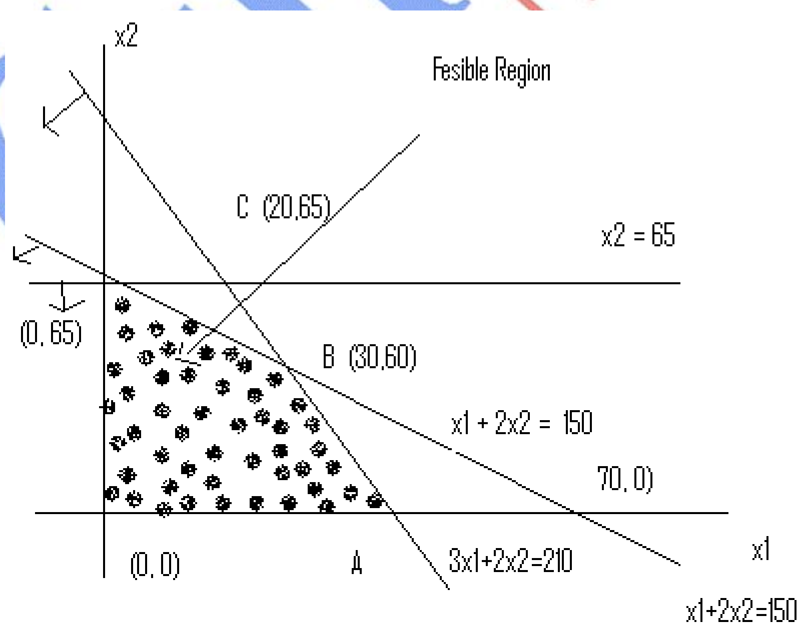
$$\left.\begin{array}{l} Z = 20x_1 + 30x_2 \\ \text{Subject to} \quad 3x_1 + 2x_2 \leq 210 \\ \qquad\qquad\quad 2x_1 + 4x_2 \leq 300 \\ x_2 \leq 65 \\ \text{and } x_1 \geq 0,\ x_2 \geq 0 \end{array}\right\}$$

Graphical solution – Let us draw the boundary lines : $3x_1 + 2x_2 = 210$ or $\frac{x1}{70} + \frac{x2}{105} = 1$ ----(1)

$2x_1 + 4x_2 = 300$ or $\frac{x1}{150} + \frac{x2}{75} = 1$ ----(2)

$x_2 = 65$ – (3) $\quad x_1 = 0$ (4) $\quad x_2 = 0$- (5)

in a two dimensional space which correspond to the inequalities of the constraints.



on considering the solution space for each constraint, we find that the shaded area OABCD in their common solution space.

Further we find that the vertices of this polygon OABCD are

0 (0,0); A (70,0); B(30,60); C (20,65); D (0,65)

The value of the objective function at these corner points are given by

at A (70,0)	Z= 20 × 70+30 × 0 = 1400
at B (30,60)	Z= 20 × 30+30 × 60= 2400
at C (20,65)	Z= 20 × 20+30 × 65= 2350
at D (0,65)	Z= 20 × 0+30 × 65 = 1950

Since Z is maximum at one of the corner points of the feasible region, so we find that it is maximum at B where x_1 =30 and x_2 =60. Hence the firm should manufacture 30 items of type A and 60 items of the type B to gave the maximum profit which is equal to Rs. 2400.

Q. 4 A factory uses three different resources for the manufacture of two different products, 20 units of the resource A, 12 unit of B and 16 units of C being available . One unit of the first product requires 2,2 and 4 units of the respective resources and one unit of the second product require 4,2 and 0 unit of the respective resources. It is known that the first product gives a profit of 2 monetary units per unit and the second 3. Formulate the LPP. How many units

of each product should be manufactured, for maximizing the profit ? Solve its graphically.

Sol. The following table shows the given Data-

Resources	Products		Available Units
	P_1	P_2	
A	2	4	10
B	2	2	12
C	4	0	16
Profit per Unit	2	3	

Mathematical formulation –

Let x units of first product P_1 and y units of second product P_2 should be manufactured for maximizing the profit. so the total profit is given by Z = $2x+3y$

Since one unit of the first product P_1 requires 2,2 and 4 units, one unit of the second product P2 requires 4,2 and 0 units of the respective resources and the available units of the three resources A,B,C are 20, 12 and 16 respectively so we have the following constraints

$2x+4y \leq 20$ or $x+2y \leq 10$ - (1)

$2x+2y \leq 12$ or $x+y \leq 6$ - (2)

$4x+0y \leq 16$ or $x \leq 4$ - (3)

and $x \geq 0$, $y \geq 0$

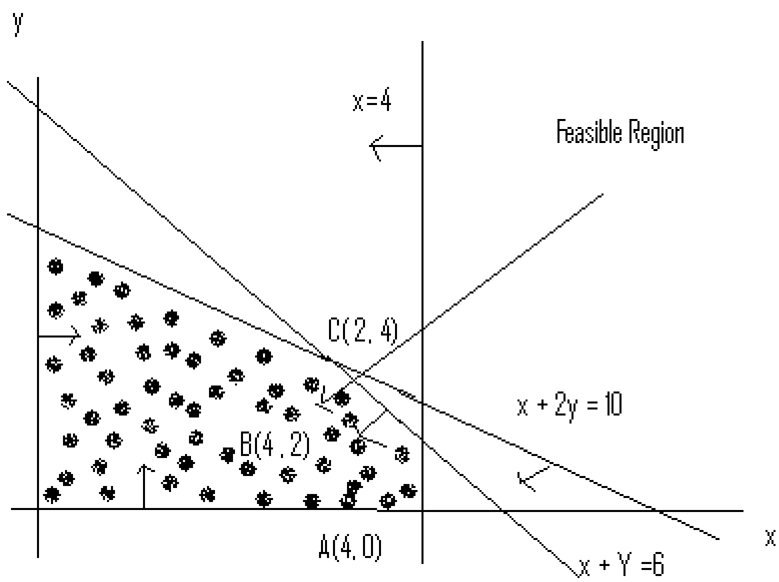
maximize $x=2x+3y$ (objective function)

subject to $x+2y \leq 10$

$x+y \leq 6$

$x \leq 4$ (constraints)

and $x \geq 0$, and $y \geq 0$,



First we draw the bounding lines

x+2y = 10; x+y = 6; x = 4; x = 0; y =0

which correspond to the inequalities of the given constraints.

Now on considering the solution space for each of the given in equality, we find that the feasible region, i.e. their common solution space is given by the shaded area OABCD. Every point of this region gives a feasible solution of the problem, where as its optimal solution is attained at one of the vertices of the area OABCD. coordinates of the four vertices are : A(4,0) ; B(4,2); C(2,4) ; D(0,5)

So we find that

at A	Z= 2×4+3×0 =8
at B	Z= 2×4+3×2 =14
at C	Z= 2×2+3×4 =16
at D	Z= 2×0+3×5 =15

Thus Z is maximum at C where x=2 and y=4. Hence 2 units of first product and 4 units of second product should be manufactured to get the maximum profit which is equal to 16 monetary units.

Q. 5 Find an optimal solution of the following L.P.P without using the simplex method .

Max f(x) = $2x_1+3x_2+4x_3+7x_4$

S.t. $2x_1+3x_2-x_3+4x_4 = 8$

$x_1-2x_2+6x_3-7x_4 = -3$

$x_1, x_2, x_3, x_4 \geq 0$

Sol. Since we know that objective function attains its optimal value at one of the extreme point and extreme points correspond to basic feasible solutions. Therefore first we find all basic feasible solutions and then we shall decide which of them gives the optimal solution.

The given system of equation can be expressed as

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = b$

Where $\alpha_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $\alpha_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$; $\alpha_3 = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

$\alpha_4 = \begin{pmatrix} 4 \\ -7 \end{pmatrix}$ and $b = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

Here n, the number of variable =4 and

m, the number of equation =2

therefore basic feasible solution $4C_2 = 6$

Now six sets of two vectors out of four vectors α_1, α_2, α_3, α_4 are

$B_1 = [\alpha_1, \alpha_2]$; $B_2 = [\alpha_1\ \alpha_3]$; $B_3 = [\alpha_1\ \alpha_4]$; $B_4 = [\alpha_2\ \alpha_3]$; $B_5 = [\alpha_2 \alpha_4]$; $B_6 = [\alpha_3\ \alpha_4]$;

Now $|B_1| = -7$; $|B_2| = 13$; $|B_3| = -18$;

$|B_4| = 16$; $|B_5| = -13$; $|B_6| = -17$

since none of these vanishes, so every set of two vectors of A are linearly Independent. Hence all the basic solution exist.

Also we know that the basic solutions corresponding to the metrix B is given by

$X_B = B^{-1}b$ so we have

$X_{B1} = \begin{bmatrix} x1 \\ x2 \end{bmatrix} = B_1^{-1}b = \frac{-1}{7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$X_{B2} = \begin{bmatrix} x1 \\ x3 \end{bmatrix} = B_2^{-1}b = \frac{1}{13}\begin{bmatrix} 6 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 45/13 \\ -14/13 \end{bmatrix}$

$X_{B3} = \begin{pmatrix} x1 \\ x4 \end{pmatrix} = B_3^{-1}b = \frac{-1}{18}\begin{bmatrix} -7 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 22/19 \\ 7/9 \end{bmatrix}$

$X_{B4} = \begin{pmatrix} x2 \\ x3 \end{pmatrix} = B_4^{-1}b = \frac{-1}{16}\begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 45/16 \\ 7/16 \end{bmatrix}$

$X_{B5} = \begin{bmatrix} x2 \\ x4 \end{bmatrix} = B_5^{-1}b = \frac{-1}{13}\begin{bmatrix} -7 & -4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 44/13 \\ -7/13 \end{bmatrix}$

$X_{B6} = \begin{bmatrix} x3 \\ x4 \end{bmatrix} = B_6^{-1}b = \frac{1}{18}\begin{bmatrix} -7 & -4 \\ -6 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 44/17 \\ 45/17 \end{bmatrix}$

out of these six basic solution X_{B2}, X_{B5} do not satisfy the non-negative restriction and hence they will not give the basic feasible solutions.

Thus basic feasible solutions are

X_1= (1,2,0,0); X_2 = (22/9,0,0, 7/9);

X_3 (0,45/16, 7/16,0); X_4 = (0,0, 44/17, 45/17)

The values of the objective function f(x) corresponding to these solution are

$f(X_1) = 2\times1+3\times2+4\times0+7\times0= 8$

$f(X_2) = 2\times\frac{22}{9}+3\times0+4\times0+7\times\frac{7}{9}= \frac{93}{9}$

$f(X_3) = 2\times0+3\times\frac{45}{16}+4\times\frac{7}{16} +7\times0= \frac{163}{16}$

$f(X_4) = 2\times0+3\times0+4\times\frac{44}{17} +7\times\frac{45}{17}= \frac{491}{17}$

Since $f(X_4)$ gives the maximum value an optimal solution to the given L.P.P.

x_1=0, x_2=0, $x_3=\frac{44}{17}$, $x_4= \frac{45}{17}$ and the maximum value of the objective function is $\frac{491}{17}$

Q. 6 Solve the following L.P.P.

Max. Z = 5 x_1+3 x_2

S.t. 3 x_1+5 x_2≤15

5 x_1+2 x_2≤10

and x_1, x_2≥0

Sol. First we write problem in standard form for simplex method by adding two slack variables x_3 and x_4 . The problem in its new form is

Max. Z= 5 x_1+3 x_2 0. x_3+ 0. x_4

S.t. 3 x_1+5 x_2+1. x_3 +0. x_4=15

5 x_1+2 x_2+0. x_3 +1. x_4=10

and xj ≥ 0, j=1,2,3,4

The coefficient matrix A is given by

$$A = \begin{bmatrix} 3 & 5 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{bmatrix} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

Taking the initial base B= (α_3 , α_4) the first simplex table is constructed as follows;

First Simplex Table

			Cj→	5	3	0	0
CB	B	XB	b	y1	y2	y3	y4
O	α_3	x_3	15	3	5	1	0
O	α_4	x_4	10	[5]	2	0	1
	Zj-	Cj	→	-5↑	-3	0	0↓

Since Z_1-C_1, =-5 is the most negative so α_1 is the entering vector also

min $\left\{\frac{x\,Bi}{yi1},\ Yi1>0\right\}$= Min $\left\{\frac{15}{4},\frac{10}{5}\right\}$ = $\frac{10}{5}$= $\frac{xB2}{y21}$

∴ $y_2$1, i.e. , 5 is the key element and the departing vector is β_2, i.e. second vector of the basic which is α_4

Now using the usual transformations we obtain the next simplex table as follows :

Second simplex table

			Cj→	5	3	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
0	α_3	x_3	9	0	$\boxed{19/5}$	1	-3/5
5	α_1	x_1	2	1	2/5	0	-1/5
	Zj	Cj→		0	-1↑	0↓	1

The basic feasible solution given by this table is not optimal, so we proceed for a new improved basic feasible solution (B.F.S.)

Since Z_2-C_2 = -1 is the most negative so α_2 is the entering vector . Also

min $\left\{\frac{x\,Bi}{yi2},\ Yi2>0\right\}$= Min $\left\{\frac{45}{19}, \frac{10}{2}\right\} = \frac{45}{19} = \frac{xB1}{y12}$

∴ the key element is y12 = 19/5 and the departing vector is β_1, i.e., first vector of the basic which is α_3, So the next simplex table is obtained as

Third Simplex Table

			Cj→	5	3	0	0
C_B	B	X_B	b	y_1	y_2	y_5	y_6
3	α_2	x_2	45/19	0	1	5/19	-3/19
5	α_1	x_1	20/19	1	0	-2/19	5/19
		Zj-Cj	→	0	0	5/19	16/19

Since all Z_j-C_j ≥0, the solution at this iteration is optimal

Thus here $x_2 = \frac{45}{19}$, $x_1 = \frac{20}{19}$, x_3 =0, x_4 =0

Hence the solution of the original problem is

$x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

And the maximum Z = $5\times\frac{20}{19} + 3\times\frac{45}{19} = \frac{235}{19}$

Q. 7 Solve the following L.P.P

Max. Z = $2x_1+5x_2+7x_3$

St. $3x_1+2x_2+4x_3 \le 100$

$x_1+4x_2+2x_3 \le 100$

$x_1+x_2+3x_3 \le 100$

and $x_1, x_2, x_3 \ge 0$

Sol. First we convert the problem in its standard form of simplex method by adding x_4, x_5, x_6 as three slack variables. The problem in its new form is

Max Z = $2x_1+5x_2+7x_3+0.x_4+0.x_5+0.x_6$

$3x_1+2x_2+4x_3+1.x_4+0.x_5+0.x_6=100$

xx $x_1+4x_2+2x_3+0.x_4+1.x_5+0.x_6=100$

$x_1+1.x_2+3x_3+2x_3+0.x_4+0.x_5+1.x_6=100$

and $x_j \ge 0$, j = 1,2,3,4,5,6

The coefficient matrix A is given by

$$A = \begin{bmatrix} 3 & 2 & 4 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{bmatrix} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$$

Taking the initial base B = $(\alpha_4, \alpha_5, \alpha_6)$ the first simplex table is constructed as follows :

First Simplex Table

			Cj→	2	5	7	0	0	0
CB	B	X_B	b	y1	y2	y3	y4	y5	y6
0	α_4	x_4	100	3	2	[4]	1	0	0
0	α_5	x_5	100	1	4	2	0	1	0
0	α_6	x_6	100	1	1	3	0	0	1
		Zj- Cj	→	-2	-5	-7	0	0	0
						↑	↓		

Since Z_3-C_3 =-7 is the most negative, So α_3 is the entering vector. Also we find that min

$$\min\left\{\frac{x_{Bi}}{y_{i3}},\ Y_{i3}>0\right\}= \text{Min}\left\{\frac{100}{4},\frac{100}{2},\frac{100}{2}\right\} = \frac{100}{4}= \frac{x_{B1}}{y_{13}}$$

which occurs in the first row so β i.e α_4 is the departing vector and 4 is the key element.

Now with the help of usual transformations we construct the next simplex table as follows.

Second Simplex Table

			Cj→	2	5	7	0	0	0
CB	B	X_B	b	y1	y2	y3	y4	y5	y6
7	α_3	x_3	25	3/4	1/2	1	1/4	0	0
0	α_5	x_5	50	-1/2	[3]	0	-1/2	1	0
0	α_6	x_6	25	-5/4	-1/2	0	-3/4	0	1
			Zj- Cj →	13/4	-3/2	0	7/4	0	0

↑ ↓

The B.F.S. given by second simplex table is not optimal so we proceed for a new improved B.F.S. For preparing next simplex table we observe that α_2 is the entering vector, α_5 is the departing vector and 3 is the key element. The next table is as follows :

Third Simplex Table

			Cj→	2	5	7	0	0	0
CB	B	X_B	b	y1	y2	y3	y4	y5	y6
7	α_3	x_3	50/3	5/6	0	1	1/3	-1/6	0
5	α_2	x_2	50/3	-1/6	1	0	-1/6	1/3	0
0	α_6	x_6	100/3	-4/3	0	0	-5/6	1/6	1
			Zj- Cj →	3	0	0	3/2	1/2	0

Since all the value of Zj-Cj ≥0 So the solution given by this table is optimal and it is given by x_1 =0, x_2 =50/3, x_3 =50/3, x_4 =0, x_5 =0, x_6 =100/3

Hence for the original problem the optimal

solution is x_1 =0, $x_2 =, \frac{50}{3}$, $x_3 =\frac{50}{3}$

and the maximum = Z= $2\times0+5\times\frac{50}{3}+7\times\frac{50}{3}$=200

Q. 8 Solve the following L.P.P

Min Z = x_1= 3 x_2 +2 x_3

S.t. 3x_1- x_2 +3$x_3 \leq 7$

-2 x_1 +4 $x_2 \leq 12$

-4x_1 +3x_2 +8 $x_3 \leq 10$

and x_1, x_2, $x_3 \leq 0$

Sol. This is a problem of minimization so converting the objective function for maximization, we have max. (-Z) = x_1+3 x_2 -2 x_3

or max. (-Z)' = - x_1+3 x_2-2x_3 where –Z =Z[1]

Now to express the constraints in the form of equation we introduce x_4 , x_5, x_6 as three slack variables and then the problem takes the following standard form for simplex method.

Max Z' = - x_1 +3x_2-2x_3+0. x_4+0. x_5+0. x_6

3x_1- x_2 +3x_3+1. x_4+0. x_5+0. x_6=7

-2x_1+4x_2 +0.x_3+0. x_3+0. x_4+1. x_5+0. x_6=12

-4x_1+3x_2 +8.x_3+0. x_4+0. x_4+0. x_5+1. x_6=10

and xj≥0, j = 1,2,3,4,5,6

for this, system of constraints is

$$A = \begin{bmatrix} 3 & -1 & 3 & 1 & 0 & 0 \\ -2 & 4 & 0 & 0 & 1 & 0 \\ -4 & 3 & 8 & 0 & 0 & 1 \end{bmatrix} (\alpha_1 , \alpha_2 , \alpha_3 , \alpha_4 , \alpha_5, \alpha_6)$$

Taking the initial base B = (α_4 , α_5, α_6) the first simplex table is constructed as follows.

First Simplex Table

			Cj→	2	5	7	0	0	0
CB	B	X_B	b	y1	y2	y3	y4	y5	y6
0	α_4	x_4	7	3	-1	3	1	0	0
0	α_5	x_5	12	-2	[4]	0	0	1	0
0	α_6	x_6	10	-4	3	8	0	0	1
		Zj- Cj →		1	-3	2	0	0	0
					↑	↓			

Since Z_2-C_2 = -3 is the most negative, So α_2 is the entering vector. Also we find that min $\left\{\frac{x\ Bi}{yi2},\ Yi2>0\right\}$ =min $\left\{\frac{12}{4}, \frac{10}{3}\right\} = \frac{12}{4}, \frac{x\ B2}{y22},$

∴ The departing vector is β_2 , i.e. the second vector of the base which α_5. Now we construct the second simplex table using the usual transformation.

Second Simplex Table

			Cj→	2	5	7	0	0	0
C_B	B	X_B	b	y1	y2	y3	y4	y5	y6
0	α_4	x_4	10	[5/2]	0	3	1	1/4	0
3	α_2	x_2	3	-1/2	1	0	0	1/4	1
0	α_6	x_6	1	-5/2	0	8	0	-3/4	0
		Zj- Cj →		-1/2	0	2	0	3/4	0
			↓						

From this table we find that the basic feasible solution B.F.S. given by it is not optimal. so we proceed for a new improved B.F.S.Zj-Cj = 1/2 is the most negative, so α_1

is the entering vector. Also min $\left\{\frac{10}{5/2}\right\}$=4 which occur in first row, α_4 is the departing vector and 5/2 is the key element. The next simplex table is as follow :

Third Simplex Table

			Cj→	-1	3	-2	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6
-1	α_1	x_1	4	1	0	6/5	2/5	1/10	0
3	α_2	x_2	5	0	1	3/5	1/5	6/20	0
0	α_6	x_6	11	0	0	-11	1	-1/2	1
		Zj-Cj	→	0	0	13/5	1/5	16/20	0

We find that all Zj-Cj ≥ 0, So the solution at this iteration is optimal.

So we have

$_1$=4 ,$_2$ =5, ,$_6$=11 ,$_3$ =0, $_5$ =0, $_6$ =11

Hence for the original problem the optimal solution is $_1$=4, $_2$=5, $_3$=0

And the maximum Z' =-4 +3×5-2×0 =11

Min Z = -11

Q.9 Solve the following L.P.P. by simplex method.

min z= x_1+ x_2

S.t. 2 x_1 +x_2 ≥4

x_1 + x_2 ≥7

and x_1, x_2 ≥0

Sol. First we convert the problem to the maximization problem by taking the objective function Z* = -Z =- x_1- x_2

Introducing the surplus variables $_3$ and $_4$ the constraints are converted into the following equations :

$2x_1 + x_2 - x_3 = 4$

$x_1 + 7x_2 - x_4 = 7$

Here we cannot get the initial B.F.S. So introduce the artificial variables $_5$ and $_6$ and assign the large negative price –M to these variables. The problem may how be expressed as Max. $Z^* = - x_1 - x_2 + 0.x_3 + 0.x_4 - Mx_5 - Mx_6$

$2x_1 + x_2 - x_3 + 0.x_3 + 0.x_4 + x_5 + 0.x_6 = 4$

$x_1 + 7x_2 + 0.x_3 - 1.x_4 + 0.x_5 + x_6 = 7$

And $x_j \geq 0$, j= 1,2,3,4,5,6 For this system the coefficient matrix A is given by

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 7 & 0 & -1 & 0 & 1 \end{bmatrix} = (\alpha1, \alpha2, \alpha3, \alpha4, \alpha5, \alpha6)$$

Taking the initial base B = (α5, α6) and we construct the first simplex table as follows :

Second simplex Table

			Cj→	-1	-1	0	0	-M
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5
-M	α_5	x_5	3	[13/7]	0	-1	1/7	1
-1	α_2	x_2	1	1/7	1	0	-1/7	0

Zj-Cj → $\frac{-13}{7}M+\frac{6}{7}$ 0   M $\frac{-1}{7}M + \frac{1}{7}$ 0

↑ ↓

First simplex Table

			Cj→	-1	-1	0	0	-M	-M
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6
-M	α_5	x_5	4	2	1	-1	0	1	0
-M	α_6	x_6	7	1	[7]	0	-1	0	1

Zj-Cj → -3M+1 -8M+1 M M 0 0

↑ ↓

Since the initial B.F.S. is not optimal, so we proceed for a new improved B.F.S.

Now Z2-C2 = -8M+1 is the most negative ,

So K=2 and as such α_2 is the entering vector. Also we find that

min

$\left\{\frac{x_{Bi}}{y_{i2}},\ Y_{i2}>0\right\}$ = Min $\left\{\frac{4}{1},\frac{7}{7}\right\} = \frac{7}{7} = \frac{x_{B2}}{y_{22}}$

So the departing vector is β_2, i.e. α_6 which is the second vector of the base and the key element is y22=7 we now prepare the next simplex table with usual transformation deleting the outgoing artificial vector α_6

Second Simplex table

Again the basic feasible solution given by this table is not optimal proceeding again we find that

min $\left\{\frac{x_{Bi}}{y_{i1}},\ Y_{i1}>0\right\}$ = Min $\left\{\frac{3}{13/7},\frac{1}{1/7}\right\} = \frac{21}{13}$

α_1 is the entering vector, α_5 us the departing vector and 13/7 is the key element.

Third *simplex Table*

			Cj→	-1	-1	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
-1	α_5	x_1	21/13	1	0	-7/13	1/13
-1	α_6	x_2	10/13	0	1	1/13	-2/13
Zj-Cj	→		0	0	6/13	1/13	

The above table shows that Zj-Cj ≥0 ∀j

Hence it gives the optimal solution which is given by $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$

min Z = -Max Z* = - $\left[\frac{-21}{13} - \frac{10}{13},\right]$

min Z = $\frac{31}{13}$

Q.10 Solve the following L.P.P. by simplex method.

max. z=2x_1+ x_2

S.t. x_1 - x_2 ≤10

2x_1 - x_2 ≤40

and x_1, x_2 ≥0

Sol. After entering slack variables x_3 , x_4 the inequalities of the constraints reduce to equations and the problem takes the form

max. Z= 2x_1+ x_2 +0. x_3+0. x_4

S.t x_1- x_1+ x_3 +0. x_4=10

2x_1- x_2+0.x_3 +1. x_4=40

and X_j≥0 j=1,2,3,4

Taking the initial Base B = (α_3, α_4) the first initial B.F.S., we construct the first simplex table as follows

First simplex Table

			Cj→	2	1	0	0
C_B	B	X_B	b	y_1	y_2	$y_3(\beta_1)$	$y_4(\beta_2)$
0	α_3	x_3	10	1	-1	1	0
0	α_4	x_4	40	2	-1	0	1
		Zj-Cj	→	-2	-1	0	0
	↑			↓			

After observe this table we find that there is a vector α_2 such that $\alpha_2 \in A$ but $\alpha_2 \notin B$ and for this vector Zj-Cj = Z_2-C_2= -1<0 and all the components of y_2 are negative. Hence the solution of this L.P.P is unbounded.

Q. 11 Show that there exists no feasible solution of the following problem.

max. Z = $3x_1+2x_2$

S.t. $2x_1+x_2 \leq 2$

$3x_1+4x_2 \geq 12$

and $x_1 \geq 0, x_2 \geq 0$

Sol. Introducing Slack variable x_3, surplus variable x_4 and artificial variable x_5 in the given constraints; the given problem can be expressed as

Max. Z = $3x_1 +2 x_2+0 x_3+ 0. x_4-Mx_5$

S.t. $2x_1+ x_2+1.x_3+0. x_4+0. x_5 =2$

$3x_1+4 x_2+0.x_3- x_4+1. x_5 =12$

and $x_j \geq 0$, j=1,2,3,4,5

Where –m is a very large negative price assigned to the artificial variable x_5

Here A = $\begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & -1 & 1 \end{bmatrix}$ =(α1, α2, α3, α4, α5)

Taking the initial base B= (α3, α5) for the initial B.F.S., we prepare the initial simplex table as follows:

First simplex Table

			Cj→	3	2	0	0	-M
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5
0	α_3	x_3	2	2	[1]	1	0	0
-M	α_5	x_5	12	3	4	0	-1	0

Zj-Cj →	-3M-3	-4M-2	0	M	0

↓

The basic feasible solution of this table is not optimal proceeding for a new B.F.S. we find that α_2 is the entering vector and α_3 is the departing vector with usual transformations the new simplex table is constructed as follows :

Second simplex Table

			Cj→	3	2	0	0	-M
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5
2	α_2	x_2	2	2	1	1	0	0
-M	α_5	x_5	4	-5	0	-4	-1	1
Zj-Cj		→		5M+1	0	4M+2	M	0

We find that Zj-Cj ≥ 0 ∀j so the condition of optimality is satisfied but we observe that the optimal solution $x_2 = 2$, $x_3 = 0$, $x_4 = 0$, $x_5 = 4$ includes the artificial variable x_5 with positive value 4. This shows that the given problem has no feasible solution because the positive value of x_5 violates the second given constraints.

Q. 12 Solve the following L.P.P.

max. $Z^* = -2x_1 - x_2$

S.t. $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1, x_2 \geq 0$

After introducing x_4 a slack variable x_3 as surplus variable and x_5, x_6 a artificial variables the system of constraints becomes.

$$3x_1 + x_2 + 0\,x_3 + 0.\,x_4 + x_5 + 0.\,x_6 = 3$$

S.t. $4x_1+3x_2+x_3+0.\,x_4+0.\,x_5+x_6=6$

$x_1+2x_2+0.x_3+x_4+0.x_5+0.\,x_6=4$

assigning the large negative price –M to the artificial variables x_5 and x_6 the objective function becomes

max. Z = $-2x_1 - x_2 + 0.x_3 + 0.\,x_4 - Mx_5 - Mx_6$

First simplex Table

			Cj→	-2	-1	0	0	-M	-M
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6
-M	α_5	x_5	3	[3]	1	0	0	1	0
-M	α_6	x_6	6	4	3	-1	0	0	1
0	α_4	x_4	4	1	2	0	1	0	0
		Zj-Cj	→	-7M+2	-4M+1	M	0	0	0

↑ ↓

The B.F.S. of this table is not optimal. Proceeding for an improved B.F.S., we find that the entering vector α_1, and the departing vector is α_5 because min $\left\{\frac{3}{3}, \frac{6}{4}, \frac{4}{1}\right\}$ $= \frac{3}{3}$

Occurs in first row we now get the next transformed table as

<u>Second simplex Table</u>

			Cj→					
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_6
-2	α_1	x_1	1	1	1/3	0	0	0
-M	α_6	x_6	2	0	[5/3]	-1	0	1
0	α_4	x_4	3	0	5/3	0	1	0
		Zj-Cj	→	0	$\frac{-5}{3}M+\frac{1}{3}$	M	0	0

↓

Again the B.F.S is not optimal proceeding further for a new B.F.S. we find that α_2 is the entering vector and α_6 is the departing vector.

Third Simplex Table

			$C_j \rightarrow$	-2	-1	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
-2	α_1	x_1	3/5	1	0	1/5	0
-1	α_2	x_2	6/5	0	1	-3/5	0
0	α_4	x_4	1	0	0	1	1
(Z_j-C_j) $\rightarrow$				0	0	1/15	0

Third simplex table shows that $Z_j-C_j \geq 0$ for all values of j hence it gives the optimal solution which is given by

$x_1 = 3/5$, $x_2 = 6/5$

and max. $Z = -2\times\frac{6}{5} - \frac{6}{5} = \frac{-12}{5}$

The sphere, The cone, Cylinder

Sphere

Definition :- A sphere is the locus of a point in space which remains at constant distance from a fixed point.

The fixed point is called the centre and the constant distance the radius of the sphere.

Equation of a sphere (Central form)

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

Where (a,b,c) are centre of sphere and radius is r

Equation of a sphere with origin as centre

$x^2+y^2+ z^2 = r^2$

General Equation of a sphere –

$x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d = 0$

where centre $(-u - v, -w)$

radius $\sqrt{(u^2 + v^2 + w^2 - d)}$

Case –I if $u^2+v^2+w^2-d=0 \rightarrow$ point sphere

if $u^2+v^2+w^2-d<0 \rightarrow$ virtual sphere

any sphere through a given circle –

$x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d + \lambda (Ax + By + Cz + D) = 0$

i.e. $S+\lambda P$ is any sphere through

the circle $S= x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d = 0$

$P= A x+By+Cz+D=0$

Equation of sphere (Diameter form)

$(x - x_1)(x - x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$

The tangent line – A line is said to be a tangent line to the sphere if it intersects a sphere in two coincident points.

The tangent plane – The locus of all tangent lines at a point on a sphere is called a tangent plane at that point.

Equation of the tangent plane at any point(α,β,γ) of the sphere

$x^2 + y^2 + z^2 + 24x + 2vy + 2wz + d = 0$

is $\alpha x + \beta y + \gamma z + u(x+\alpha) + v(y+\beta) + W(Z+\gamma) + d = 0$

Orthogonal spheres –

Two spheres are said to be orthogonal when the angle between them is a right angle.

Condition of orthogonality of two sphere

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

<u>Cone</u>

Definition – A cone is a surface generated by a straight line which passes through a **fixed point** and is subjected to one more condition. e.g. intersects a curve or touches a given sphere.

The fixed point is called the vertex of the cone and the given curve on surface is called the guiding curve or the guiding surface of the cone and any individual line on the surface of a cone is called its generator.

Equation of a cone →whose vertex is (α,β,γ) and guiding curve are given

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, $z = 0$, is expressed by –

$a(\alpha z - \gamma x)^2 + 2h(\alpha z - \gamma x)(\beta z - \gamma y) + b(\beta z - \gamma y)^2 + 2g(dz - \gamma x)(z - \gamma) + 2f(\beta z - \gamma y)(z - \gamma) + c(z - \gamma)^2 = 0$

Enveloping cone

Definition – The locus of tangent lines drawn from a given point to a given surface is called an enveloping cone or tangent cone.

Equation of the enveloping cone of the sphere

$x^2 + y^2 + z^2 = a^2$ with vertex at the point (α,β,γ) are given by $SS_1 = T^2$

i.e. $(x^2 + y^2 + z^2 - a^2)(\alpha^2+\beta^2+\gamma^2-a^2) = (\alpha x+ \beta y+ \gamma z-a^2)$

<u>Cylinder</u>

<u>Definition</u> – A Cylinder is a surface generated by a variable straight line which moves in such a way that it is always parallel to a fixed line and which fulfills one more condition e.g. It may intersect a given curve or touch a given surface or may be at a constant distance from a fixed straight line.

The fixed line is called axis or guiding line, The given curve or the surface is called the guiding curve or the guiding surface and the variable line which generates the surface of the cylinder is called the generator of the cylinder.

Equation of a cylinder – whose generators are parallel to line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

and whose guiding curve is the conic $ax^2+2hxy+by^2+2gx+2fy+c =0, z=0$

then equation – $a(nx-lz)^2 +2h(nx-lz)(ny-mz)+b(ny-mz)^2 +2gn(nx-lz)+2fn(ny-mz) +n^2c=0$

Represent the equation of a cylinder

Right circular cylinder

Definition – A right circular cylinder is the surface generated by a straight line which intersects a fixed circle and is perpendicular to its plane.

The fixed circle is called the guiding circle of the cylinder

Equation of right circular cylinder is expressed by –

$$(x-\alpha)^2 + (y-\beta)^2+(z-\gamma)^2 = r^2+ \left[\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}\right]^2$$

whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and whose radius is r

Q. 1 Find the centre and radius of the sphere

$7x^2+7y^2+7z^2-6x-3y-2z=0$,

Sol. The given equation of sphere can be written as

$$x^2+y^2+z^2-\frac{6}{7}x-\frac{3}{7}y-\frac{2}{7}z=0$$

here $u=\frac{-3}{7}$, $v=\frac{-3}{14}$, $w=\frac{-1}{7}$ $d=0$

$\therefore$ centre $\left(\frac{3}{7}, \frac{3}{14}, \frac{1}{7}\right)$ and radius =

$\sqrt{(((-3)/7)^2+((-3)/14)^2+((-1)/7)^2-0)}=1/2$

Q. 2 A sphere of constant radius r passes through the origin O and cuts the axes in A,B,C. Prove that the locus of the feet of the perpendicular drawn from the 0 to the plane ABC is given by

$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$

Sol. Suppose coordinates of A (a,0,0), B (0,b,0) C (0,0,c) then equation of sphere OABC can be written as $x^2+y^2+z^2-ax-by-cz)=0$ -(1)

According to question radius of sphere is equal to r therefore

$(a/2)^2+(b/2)^2+(c/2)^2=r^2$

or $a^2+b^2+c^2=4r^2$ -(2)

again equation of plane ABC will be

$\left(\frac{x}{a}\right)+\left(\frac{y}{b}\right)+\left(\frac{z}{c}\right)=1$ -(3)

Equation of perpendicular line on plane (3) from origin will be

$\frac{x}{1/a}=\frac{y}{1/b}=\frac{z}{1/c}=k$ (suppose) – (4)

on line (4) co-ordinates of any point will be $\left(\frac{k}{a},\frac{k}{b},\frac{k}{c},\right)$ and if we assume co-ordinates of the feet of the perpendicular are

(f,g,h) then $f=\frac{k}{a}$, $g=\frac{k}{b}$, $h=\frac{k}{c}$

$\Rightarrow a=k/f$, $b=k/g$, $c=k/h$- (5)

substitute the value of a,b,c in equation (2) we obtain

$\frac{k^2}{f^2} + \frac{k^2}{g^2} + \frac{k^2}{h^2} = 4r^2$

$\Rightarrow K^2(f^{-2}+g^{-2}+h^{-2}) = 4r^2$ (6)

again feet of the perpendicular (f,g,h) is situated on plane (3) therefore

$\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 1$

or $\frac{f^2}{k} + \frac{g^2}{k} + \frac{h^2}{k} = 1$ (after substitute the value of a,b,c)

or $\frac{1}{k}(f^2+g^2+h^2)=1$ Now eliminate the value of k from equation (6) and (7) we obtain

$(f^2+g^2+h^2)(f^{-2}+g^{-2}+h^{-2}) = 4r^2$

Q.3 Find the equation of the sphere whose centre is the point (1,2,3) and which touches the plane $3x$+2y+z+4 =0. Find also the radius of the circle in which the sphere is cut by the plane x+y+z=0

Sol. Centre of sphere (1,2,3)

Equation of sphere

$x^2 + y^2 + z^2 + x + 2y + 3z + d = 0$ - 1

If the plane touch the sphere then the length of perpendicular on this plane from centre of the sphere is equal to the radius of sphere plane.

$3x+2y+z+4=0$

$\frac{1 \times 3 + 2 \times 2 + 3 \times 1 + 4}{\sqrt{9+4+1}} = \sqrt{1+4+9-d}$

$\frac{3+4+3+4}{\sqrt{14}} = \sqrt{14-d}$

$\Rightarrow \frac{14}{\sqrt{14}} = \sqrt{14-d}$

taking square both side

$\frac{14\times14}{14}$ = 14-d or 14 = 14-d ∴ d=0

Putting the value of d in equation (1)

$x^2 + y^2 + z^2 - 2x - 4y + 6z = 0$

now radius of sphere $\sqrt{1+4+9} = \sqrt{14}$

Length of perpendicular from centre of the sphere

P = $\frac{1.1 + 2.1 + 3.1}{\sqrt{1+1+1}}$

= $\frac{6}{\sqrt{3}}$

Radius = $\sqrt{r^2 - p^2}$ where r is radius of sphere and p is perpendicular length from centre of the sphere .

= $\sqrt{(\sqrt{14})^2 - (6/\sqrt{3})^2} = \sqrt{14-12} = \sqrt{2}$

Q.4 **Find the equations of the two tangent planes to the sphere $x^2 + y^2 + z^2 = 16$ which pass through the line $x + y = 5, x - 2z = 7$**

Sol. Equation of plane which passes through the lines

$(x+y-5) + \lambda(x - 2z - 7) = 0$

$x(1+\lambda) + y - 2\lambda z - (5+7\lambda) = 0$

if plane will be a tangent plane to the sphere then length of perpendicular on this plane from centre of the sphere is equal to radius of sphere.

Centre of sphere = (0,0,0)

radius = $\sqrt{16}$

$$\frac{0\times(1+\lambda)+0.(1)+0\,(-2\lambda)-(5+7\lambda)}{\sqrt{(1+\lambda)^2+(1)^2+(-2\lambda)^2}}=\sqrt{16}$$

$$=\frac{-(5+7\lambda)}{\sqrt{1+\lambda^2+2\lambda+1+4\lambda^2}} \quad = \sqrt{16}$$

$$\frac{-(5+7\lambda)}{\sqrt{5\lambda^2+2\lambda+2}} \quad = \sqrt{16}$$

$$-(5+7\lambda) = \sqrt{16}\ \sqrt{5\lambda^2+2\lambda+2}$$

taking square on both side we obtain

$(-5+7\lambda)^2 = 16\,(5\lambda^2 + 2\lambda + 2)$

or $25+49\lambda^2+70\lambda = 80\lambda^2+32\lambda+32$

or $31\lambda^2 - 38\lambda + 7 = 0$

$\Rightarrow (\lambda-1)(31\lambda-7) = 0$

$\lambda = 1, 7/31$ substitute the value of λ in equation (1)

we obtain $2x+y-2z-12=0$

or $38x+31y-14z-204=0$

Q.5 **Find the equations of the sphere passing through (1,-1,0) and touching the following sphere at (1,2,-2);**

$x^2+y^2+z^2+2x-6y+1=0$

Sol. Equation of tangent plane which touches the sphere at (1,2,-2)

$x(1)+y(2)+z(-2)+1(x+1)-3(y+2)+1=0$

$\Rightarrow x+2y-2z+x+1-3y-6+1=0$

$2x-y-2z-4=0$

Equation of sphere

$(x^2+y^2+z^2+2x-6y+1)+\lambda(2x-y-2z-4)=0$ (1)

The sphere passes through points (1,-1,0)

$(1+1+0+2+6+1)+\lambda(3-4)$

$\Rightarrow \lambda$ =11 putting the value of λ in equation (1) we obtain

$(x^2 + y^2 + z^2 + 2x - 6y + 1) + 11\,(2x - y - 2z - 4) = 0$

$x^2 + y^2 + z^2 + 2x - 6y + 1 + 22x - 11y - 22z - 44 = 0$

$x^2 + y^2 + z^2 + 24x - 17y - 22z - 43 = 0$

Q.6 **Tangent plane at any point of sphere $x^2 + y^2 + z^2 = r^2$ meets the co-ordinate axes at A,B,C show that the locus of the point of intersection of planes drawn parallel to the co-ordinate planes through A,B,C is the surface $x^{-2} + y^{-2} + z^{-2} = r^{-2}$**

Sol. Here equation of sphere is given $x^2 + y^2 + z^2 = r^2$ and A (a,o,o) B (0,b,0) C (0,0,c)

Equation of plane will be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Point of x=a, y=b, z=c

Let the intersection of plane is (f,g,h) = (a,b,c)

$$r = \frac{\frac{1}{a}.0 + \frac{1}{b}.0 + \frac{1}{c}.0}{\sqrt{\frac{1}{a^2}. + \frac{1}{b^2}. + \frac{1}{c^2}.}} \quad \text{---1}$$

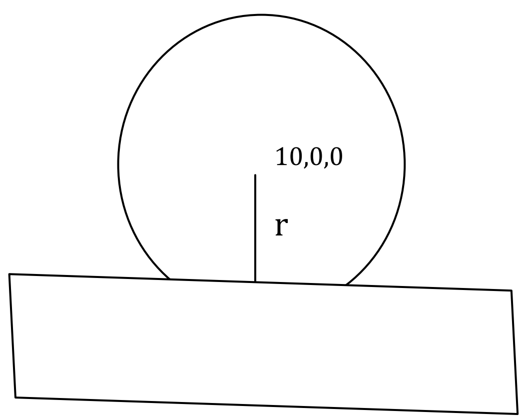
$$R = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.}}$$

$$\Rightarrow \quad \frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

a = x, b=y, c=z

$$\therefore \quad \frac{1}{r^2} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$

$$\Rightarrow \quad x^{-2} + y^{-2} + z^{-2} = r^{-2}$$



Q.7 **Find the equation of the sphere passing through the following circles and the points mentioned against then.**

(a) $x^2+y^2+z^2 = 9, \quad 2x+3y+4z=5; (1,2,3)$

Sol. Here Equation of sphere $S+\lambda P=0$

$(x^2+y^2+z^2-9)+\lambda(2x+3y+4z-5)=0$ (1)

Its passing through the points (1,2,3)

$(1+4+9-9)+\lambda(2+6+12-5)=0$

$\Rightarrow 5+15\lambda=0 \Rightarrow \lambda = -1/3$

substitute the value of λ in Eq^{in} (1) we obtain

$(x^2+y^2+z^2-9)-1/3\ (2x+3y+4z-5)=0$

$3x^2+3y^2+3z^2-2x-3y-4z-22=0$

(b) $x^2+y^2+z^2=1,\ x+2y+3z=4 \quad (0,0,0)$

Here Equation of sphere $S+\lambda P=0$

$(x^2+y^2+z^2-1)+\lambda(x+2y+3z-4)=0$ (1)

its passing through the points (0,0,0)

$-1+\lambda(-4)=0$

$\Rightarrow \alpha = -1/4$

substitute the value of λ in Eq^{in} (1) we obtain

$(x^2+y^2+z^2-1)\frac{-1}{4}(x+2y+3z-4)=0$

$4(x^2+y^2+z^2-1)-x-2y-3z+4=0$

$4(x^2+y^2+z^2)-x-2y-3z=0$

Q.8 **Find the equation to the sphere which passes through $x^2+y^2+z^2=4$, z=0, its cut by the plane**

Sol $x+2y+2z=0$ in a circle of radius 3

Equation of circle $x^2+y^2+z^2-4=0; Z=0$

Equation of sphere $S+\lambda P=0$

$(x^2+y^2+z^2-4)+\lambda z=0$ – (1)

its cut by the plane x+2y+2z =0

Centre of Sphere (0,0- $\lambda/2$)

radius of sphere $\sqrt{0+0+\frac{\lambda^2}{4}+4}$

length of perpendicular passes through from centre to plane

p = $\frac{0.1+0.2+2(-\lambda/2)}{\sqrt{1+4+4}}$ $= \frac{-\lambda}{3}$

radius of circle = $\sqrt{r^2-p^2}$

$9= r^2-p^2$

$9 = \frac{\lambda^2}{4} + 4 - \frac{\lambda^2}{9} \Rightarrow 5 = \frac{5\lambda^2}{36}$

$\Rightarrow \lambda^2=36 \qquad \Rightarrow \lambda = \pm 6$

substitute the value of λ in Eq in (1)

$(x^2+y^2+z^2-4) \pm 6z = 0$

Q.9 Find the equation of the sphere having the circle ($x^2+y^2+z^2$ = 16, $2x$-3y+6z -7) as a great circle. also find the radius and centre of the great circle.

Sol. Any sphere through the given circle is

$(x^2+y^2+z^2-16) + \lambda(2x-3y+6z-7)=0$ (1)

Centre of sphere $(-\lambda, -3/2\lambda, -3\lambda)$

d= -(16+7λ)

if the given circle is a great circle of (1) the centre $(-\lambda, \frac{-3}{2}\lambda, -3\lambda)$ lie on the plane $2x$-3y+6z =7

$-2\lambda - \frac{9}{2}\lambda - 18\lambda = 7$

$\Rightarrow \frac{-49\lambda}{2} = 7 \Rightarrow \lambda = -2/7$

Substitute the value of λ in (1) we obtain $(x^2+y^2+z^2-16)\frac{-2}{7}(2x-3y+6z-7)=0$

$x^2+y^2+z^2 \frac{-1}{7}$ $(4x-6y+12z)-14=0$

point of centre $(\frac{2}{7}, \frac{-3}{7}, \frac{6}{7})$

radius of sphere = $\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}+14}$

$= \sqrt{15}$

$P=\frac{\frac{2}{7}\times 2+\frac{3}{7}\times 3+6\times\frac{6}{7}-7}{\sqrt{4+9+36}}$ $=0$

radius of circle = $\sqrt{r^2-p^2} = \sqrt{15-0}$ $= \sqrt{15}$

Equation of perpendicular line from C (2/7, -3/7, 6/7)

$\frac{x-2/7}{2} = \frac{x+3/7}{-3} = \frac{x-6/7}{6}$ = r (let)

Co-ordinate of point (2r+2/7, -3r-3/7, 6r+6/7)

2(2r+2/7) +(-3) (-3r-3/7)+6 (6r+6/7)=7

r=0

point of circle (2/7, -3/7, 6/7)

Q.10 **Find the equation of the cone whose vertex and guiding curve are as – (1,1,0) ; $x^2+z^2=4$ y =0**

Sol. Let Equation of generating line passing through the vertex (1,1,0) and whose direction ratio are l,m,n

$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-0}{n}$ - (1)

Equation (1) meet plane y=0, therefore

$\frac{x-1}{l} = \frac{-1}{m} = \frac{z}{n}$

Intersection point of plane and line is

$(1-\frac{l}{m}, o, \frac{-n}{m})$

if its point situated on guiding curve then

$(1-\frac{l}{m})^2 + \frac{n^2}{m^2} = 4$ - (2)

Eliminate the value of l,m,n from Eq^{in} (1) & (2)

$(1-\frac{x-1}{y-1})^2 + \frac{z^2}{(y-1)^2} = 4$

$(y-1-x+1)^2 + z^2 = 4\,(y-1)^2$

$x^2+3y^2+z^2-2xy+8y-4=0$

Q.11 **If the straight line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-y}{n}$ intersect the curve $ax^2+by^2=1$; z=0, then prove that $a(\alpha n-\gamma l)^2 + b\,(\beta n-\gamma m)^2 = n^2$**

Sol.

Equation of straight line

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-y}{n}$ (1)

its meet plane z=0 $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-y}{n}$

intersection point of plane and line is

$(\alpha-\frac{l\gamma}{n},\ \beta-\frac{my}{n}, 0\ )$

if its points situated on curve $ax^2+by^2=1$ then

$a\,(\alpha-\frac{l\gamma}{n})^2+b\,(\beta-\frac{my}{n})^2=1$

$a\,(\alpha n-\gamma l)^2 + b(\beta n-m\gamma)^2 = n^2$

Q.12 **Find the equation to the cone whose vertex is the origin and which passes through the curve of intersection of : $ax^2+by^2+cz^2=1$; $\alpha x^2+\beta y^2=2$**

Sol. $ax^2+by^2+cz^2=1$ -(1)

$\alpha x^2+\beta y^2=2 \Rightarrow \frac{\alpha x^2}{2}+\frac{\beta y^2}{2}=1$ -(2)

with the help of Equation (2) Equation (1) become Homogenous and by this process we obtain Equation of the cone

$ax^2+by^2+cz^2=2 \quad = \frac{\alpha x^2}{2}+\frac{\beta y^2}{2}=1$

$2(ax^2+by^2+cz^2) = \alpha x^2+\beta y^2 = 2$ Ans.

Q.13 **Prove that the lines drawn from the origin so as to touch the sphere**

$x^2 + y^2 + z^2$ +2u x +2vy+2wztd=0 lie on the cone d($x^2 + y^2 + z^2$) = (4x+vy+wz)²

Sol. Equation of line passing through the vertex (0,0,0)

$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$ =r (let) - (1)

point on line (1) is (lr, mr, nr). if its point are situated on $x^2+y^2+z^2$ +2ux+2vy+2wztd=0 then

$l^2r^2 + m^2r^2 + n^2r^2$+24lr+2vmr+2wnr+d =0

r^2 ($l^2+m^2+n^2$)+2r (ul+vm+wn) +d=0

∴ line touch the sphere ∴ B^2-4AC=0

4 (ul+vm+wn)² -4 ($l^2+m^2+n^2$).d=0

(ul+vm+wn)² = ($l^2+m^2+n^2$).d

now eliminate l,m,n we can write

(ux+vy+wz)² = ($x^2+y^2+z^2$).d Ans

Q.14 **Find the equation of the cone whose vertex is (α, β, γ) and which touches the surface $ax^2+by^2+cz^2$ =1**

Sol. Equation of line passing through the vertex (α, β, γ) is

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ =r (suppose) - (1)

the point on line (1) is $(\ell r + \alpha, mr + \beta, nr + \gamma)$

if its point situated on $ax^2+by^2+cz^2$ =1 then

$a(lr + \alpha)^2$+b $(mr + \beta)^2$+c $(nr + \gamma)^2$=1

$r^2(al^2 + bm^2 + cn^2) + 2r\,(al\alpha + bm\beta + cn\gamma)$

$+a\alpha^2 + b\beta^2 + c\gamma^2 - 1$=0

Since line touches the sphere therefore $B^2 - 4AC = 0$

4(al$\alpha + bm\beta + cn\gamma)^2$-4 $(al^2$+bm²+cn²) $(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$=0

Now Eliminate value of l,m,n we obtain

$[a(x-\alpha)\alpha + b(y-\beta)\beta + c(z-\gamma)\gamma]^2$

= $[a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2] \times \quad (a\alpha^2 + b\beta^2 + c\,\gamma^2 - 1) = 0$

$(ax\alpha + by\beta + cz\gamma = (ax^2+by^2+cz^2)(a\alpha^2 + b\beta^2 + c\gamma^2=1)$

Q.15 **Prove that the following equation represent a cone with vertex (1,-2,2)**

$7x^2+2y^2+2z^2-10zx+10xy+26x-2y+2z-17=0$

Sol. $F(x,y,z,t) = 7x^2+2y^2+2z^2-10zx+10xy+26xt$

$-2yt+2zt-17t^2=0$

$\therefore \left(\frac{\partial f}{\partial x}\right)_{t=1} = 0 \Rightarrow 14x + 10y - 10z + 26 = 0$ - (1)

$\therefore \left(\frac{\partial f}{\partial y}\right)_{t=1} = 0 \Rightarrow 4y + 10x - 2 = 0$ - (2)

$\therefore \left(\frac{\partial f}{\partial z}\right)_{t=1} = 0 \Rightarrow 4z - 10x + 2 = 0$ -(3)

$\therefore \left(\frac{\partial f}{\partial t}\right)_{t=1} = 0 \Rightarrow 26x - 2y + 2z - 34 = 0$ - (4)

Now solve (1), (2), (3) we obtain

$x = 1,\ y = -2,\ z = 2$

the value of $x = 1,\ y = -2,\ z = 2$ is satisfied by equation (4) therefore the given equation represent a cone whose vertex is (1,-2,2)

Q.16 **Prove that the angle between the lines in which the x +y+z=0 cuts a cone $a$yz+bz x+c xy=0 is**

(a) **$\frac{\pi}{2}$ 2 if a+b+c =0**

(b) **$\frac{\pi}{3}$ if 1/ a+1/b +1/c =0**

Sol. Equation of plane is x +y+z=0 (1)

Equation of cone is $a$yz+bz x+c xy=0 (2)

Let Equation of intersection line between (1) & (2)

is $\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} = r$

point on line → (lr, mr, nr) if this point situated on cone and plane then

l+m+n =0 - (3)

$a$mn+bln+clm=0 - (4)

Now eliminate the value of n from (3) & (4)

am (-l-m) +b(-l-m) l+clm=0

-$a$ml-am²- bl²-bml+clm =0

$a\frac{\ell}{m} + a$ +b $(\frac{\ell}{m})^2$ +b $(\frac{\ell}{m})$ -c $\frac{\ell}{m}$ =0

$b\left(\frac{\ell}{m}\right)^2 + \frac{\ell}{m}(a$+b-c)+a=0

$\frac{\ell_1}{m_1} + \frac{\ell_2}{m_2} = \frac{-\ (\text{a+b+c})}{b} = \frac{\ell_1}{m_1} \quad \frac{\ell_2}{m_2} = \frac{a}{b}$

$\frac{l_1\, m_2 + l_2 m_1}{m_1\, m_2} = \frac{-\ (\text{c}-a-\text{c})}{b}$

$\Rightarrow \frac{l_1\, m_2 + l_2 m_1}{(\text{c}-a-\text{c})} = \frac{-\ \text{m}_1\text{m}_2}{b} = \text{K}$

$\Rightarrow \frac{l_1\, l_2}{\text{a}} = \frac{\text{m}_1\text{m}_2}{b} = \frac{n_1\, n_2}{\text{c}} = \text{K}$

(i) $\cos\theta = l_1\, l_2 + m_1 m_2 + n_1 n_2$

$\cos\frac{\pi}{2} = a\text{k+bk+ck}$

0= a+b+c

(ii) $\tan\theta = \frac{\sqrt{\Sigma(l_1 m_2 - l_2 m_1)^2}}{l_1 l_2 + m_1 m_2 + n_1 n_2}$

$\tan\frac{II}{3} = \frac{\sqrt{\Sigma[(l_1 m_2 + l_2 m_1)^2 - 4 l_1 l_2 m_1 m_2}}{ak+bk+ck}$

$\sqrt{3} = \frac{\sqrt{\Sigma[K^2\ (c-a-b)^2 - 4akbk}}{ak+bk+ck}$

$\sqrt{3} = \frac{\sqrt{\Sigma\ [c^2+a^2+b^2-2ac-2ab-2bc]}}{a+b+c}$

$\sqrt{3} = \frac{\sqrt{3\ [a^2+b^2+c^2-2ab-2bc-2ca]}}{a+b+c}$

$\sqrt{3} = \frac{\sqrt{3\ [a^2+b^2+c^2-2ab-2bc-2ca]}}{a+b+c}$

$(a+b+c)^2 = [a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$

$$4ab + 4bc + 4ca = 0$$

$$\frac{ab+bc+ca}{abc} = 0$$

$$\therefore \frac{1}{c}+\frac{1}{a}+\frac{1}{b} = 0$$

Q.17 If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represent one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two

Sol. – Equation of perpendicular plane on line

$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is

$1x + 2y + 3z = 0$ – (1)

Equation of cone $5yz - 8zx - 3xy = 0$ - (2)

Equation of Intersection line between (1) & (2) is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r$

If its line situated on cone and plane then

$\ell + 2m + 3n = 0$ (3)

5mn-8nl-3lm =0 (4)

Now Eliminating the value of l in Eq in (3) & (4)

5mn-8n (-2m-3n) -3 (-2m-3n) m=0

$\Rightarrow 30mn + 24n^2 + 6m^2 = 0$

$\Rightarrow m^2 + 5mn + 4n^2 = 0$

$\Rightarrow (m+4n)(m+n) = 0$

$\therefore m = -4n$; $m = -n$

substitute the value of m = -4n in Eq^{in} (3)

$\ell + 2(-4n) + 3n = 0$

$\Rightarrow \ell - 5n = 0 \Rightarrow \ell = 5n$

$\frac{\ell}{5} = \frac{m}{-4} = \frac{n}{1}$ - (5)

Now Eliminate ℓ, m, n we can write

$\frac{x}{5} = \frac{y}{-4} = \frac{z}{1} =$ (6)

Now again m=-n in Eq^{in} (3) we obtain

ℓ+n =0

$\frac{\ell}{-1} = \frac{m}{-1} = \frac{n}{1}$

or $\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1}$ (7)

line (6) and (7) will be perpendicular

If $a_1a_2 + b_1b_2 + c_1c_2 = 0$

5(-1)+(-4) (-1) +1.1 =0

$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} =$ (8) [given line Eq^{in}]

Eq^{in} (8) ⊥ Eq^{in} (7)

then 1 (-1) +2 (-1) + 3(1) =0

-1-2+3 =0 Ans.

Q.18 Find the angle between the lines of section of the plane $3x + y + 5z = 0$ with the cone $6yz - 2zx + 5xy$=0

Sol. Equation of plane $3x + y + 5z = 0$ - (1)

Equation of cone $6yz - 2zx + 5xy$=0 - (2)

Equation of intersection line between cone and plane is given by

$\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ =r - (3)

If its line situated on plane and cone then

$3\ell + m + 5n = 0$ - (4)

6mn$-2nl + 5lm = 0$ - (5)

Now Eliminate the value of m from Eq^{in} (4) & (5)

6n (-3$\ell - 5n$) -2nℓ + 5ℓ (-3$\ell - 5n$)=0

⇒ -18n ℓ-30$n^2 - 2n\ell$ -15$\ell^2 - 25\ell n$ =0

⇒ ℓ^2+3ℓn+2n^2=0

ℓ= -n, ℓ= -2n

Now put ℓ= - n in Eq^{in} (4)

-3n +m+5n =0 ⇒ m+2n =0

$\Rightarrow$ m = -2n

$\frac{\ell}{-1} = \frac{m}{-2} = \frac{n}{1}$ (6)

Now again put ℓ =-2n in Eq^{in} (4)

3(-2n)+m+5n =0

$\Rightarrow$ m-n=0 $\Rightarrow$ m=n

$\frac{\ell}{-2} = \frac{m}{1} = \frac{n}{1}$ - (7)

angle between line (6) & (7)

$\cos\theta = \frac{(-1)(-2)+(-2)(1)+1(1)}{\sqrt{1+4+1}\ \sqrt{1+4+1}}$

$\cos\theta = \frac{3-2+1}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$ $\therefore 0 = \cos^{-1}\left(\frac{1}{6}\right)$

Q.19 Prove that the plane $lx + my + nz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular lines if $(b+c)\ell^2 + (c+a)m^2 + (a+b)n^2 = 0$

Sol. Equation of plane $\ell x + my + nz = 0$ - (1)

Equation of cone $ax^2 + by^2 + cz^2 = 0$ - (2)

if plane (1) intersect cone (2) then equation of intersection line between plane and cone is $\frac{x}{u} = \frac{y}{v} = \frac{z}{w} = K$

Its line situated on plane and cone therefore

$\ell u + mv + nw = 0$ (3)

$au^2 + bv^2 + cw^2 = 0$ (4)

Eliminate the value of w from (3) & (4)

$au^2 + bv^2 + c\left(\frac{-lu-mv}{n}\right)^2 = 0$

$(au^2 + bv^2)n^2 + c\ (\ell^2u^2 + m^2v^2 + 2\ell muv) = 0$

$\left(\frac{u}{v}\right)^2 (an^2 + cl^2) + 2\frac{u}{v}(\ell mc) + bn^2 + cm^2 = 0$

$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{bn^2 - cm^2}{an^2 + cl^2}$

Now eliminate the value of v from (3) & (4)

$$au^2 + b\left(\frac{-lu-nw}{m}\right)^2 + cw^2 = 0$$

$$(au^2 + cw^2)m^2 + b\;(\ell^2u^2 + n^2w^2 + 2\ell nuw) = 0$$

$$C\left(\frac{w^2}{u^2}\right)m^2 + am^2 + b\ell^2 + bn^2\frac{w^2}{u^2} + 2\ell bn\frac{w}{u} = 0$$

$$\left(\frac{w}{u}\right)^2(6m^2 + bn^2) + 2\frac{w}{u}(\ell nb) + am^2 + b\ell^2 = 0$$

$$\frac{w_1}{u_1}\;\frac{w_2}{u_2} = \frac{am^2+b\ell^2}{bn^2+cm^2}$$

$$\frac{u_1u_2}{6n^2+cm^2}\;\frac{v_1v_2}{an^2+cl^2} = \frac{w_1w_2}{am^2+bl^2} = K$$

∴ lines are perpendicular therefore

$u_1u_2 + v_1v_2 + w_1w_2 = 0$

$(bn^2+cm^2) + (an^2+cl^2) + (bl^2+am^2) = 0$

$(b+c)\,l^2 + (c+a)\,m^2 + (a+b)n^2 = 0$

Q.20 Find the equations of a cylinders whose generators are parallel to the following line and intersecting the curve mentioned against them.

$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$, ellipse $x^2 + 2y^2 = 1$ z=0

Sol.: Let P (α, β, γ) be any point on the cylinder then equations of a generator through (α, β, γ) are

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{Z-\gamma}{3}$$

This meets Z=0 plane at the point given by

$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{-\gamma}{3}$ i.e. $(\alpha\frac{-\gamma}{3}, \beta + \frac{2\gamma}{3}, 0)$

This point lies on the curve $x^2 + 2y^2 = 1$, z=0 if

$$\left(\alpha - \frac{\gamma}{3}\right)^2, + 2\left(\beta + \frac{2\gamma}{3}\right)^2 = 1$$

Hence the locus of P (α, β, γ) is

$$(x - z/3)^2 + 2\,(y + 2z/3)^2 = 1$$

$$\Rightarrow 9x^2 + 9z^2 + 18y^2 - 6\,xz + 24yz = 9$$

$\Rightarrow 3x^2 + 3z^2 + 6y^2 + 8yz - 2xz - 3 = 0$

which is the required equation of the cylinder.

Q.21 Find the equations of a cylinders whose generators are parallel to the line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ and base is the curve $x^2 + y^2 + 2gx + 2fy + c = 0,\ z = 0$

Sol. Let P (α, β, γ) be any point on the cylinder then equations of a generator through P are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ (let) – (1)

This meets z=0 plane at the point given by

$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$ i.e. $(\alpha - \frac{\ell\gamma}{n}, \beta - \frac{m\gamma}{n}, 0)$

This point lies on the curve then

$(\alpha - l\gamma/n)^2 + (\beta - m\gamma/n)^2 + 2g(\alpha - \frac{\ell\gamma}{n}), + 2f(\beta - \frac{m\gamma}{n}) + c = 0$

Hence the locus of P (α, β, γ) is

$(nx - \ell z)^2 + (ny - mz)^2 + 2gn(nx - \ell z) + 2f(ny - mz).n + cn^2 = 0$

Q.22 Find the equations of the enveloping cylinder of the surface $ax^2 + by^2 + cz^2 = 1$ and whose generator are parallel to the line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$

Sol. Let P (α, β, γ) be any point on the cylinder then equation of generator through P are

$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ (let) – (1)

the point on generator given by

$(\ell r + \alpha,\ mr + \beta,\ nr + \gamma)$

This points lies on the curve then

$a(\ell r + \alpha)^2 + b(mr + \beta)^2 + c(nr + \gamma)^2 = 1$

$(a\ell^2 + bm^2 + cn^2)r^2 + 2r(a\ell\alpha + bm\beta + cn\gamma) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$ -(2)

line (1) touches the concoid if $B^2 - 4Ac = 0$

$4(a\ell\alpha + bm\beta + cn\gamma)^2 - 4(a\ell^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$

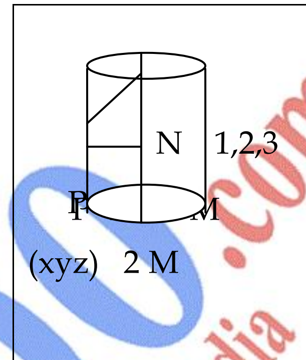
Hence the locus of P (α, β, γ) is

$(ax^2+by^2+cz^2-1)\,(a\ell^2+bm^2+cn^2)=(a\ell x+bmy+cnz)^2$

Q.23 Find the equation of a right circular cylinder whose axis and radius are $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}$; 2

Sol. Let P (x,y,z) be any point on cylinder

from right angle Δ OPM



$PN^2=PM^2+MN^2$

$(x-1)^2+(y-2)^2+(z-3)^2=4+\left[\frac{(x-1).2+(y-2).1+(z-3).2}{\sqrt{4+1+4}}\right]^2$

$\Rightarrow 9(x^2-2x+y^2-4y+z^2-6z+1+4+9)$

$=36+(2x+y+2z-10)^2$

$\Rightarrow 9x^2-18x+9y^2-36y+9z^2-54z+126$

$=36+4x^2+y^2+4z^2-40z+100+4xy-40x+8xz+4yz-20y$

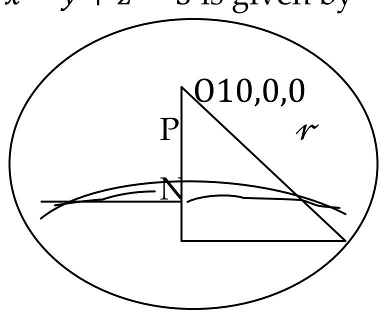
$\therefore 5x^2+8y^2+5z^2-4yz-8xz-4xy+22x-16y-14z-10=0$

which is the required cylinder

Q.24 Find the equation of the right circular cylinder whose guiding circle is $x^2+y^2+z^2=9,\ x-y+z=3$

Sol. Radius of the sphere $r=3$ and P_1 the length of the perpendicular from the centre of the sphere on the plane $x-y+z=3$ is given by

$\frac{3}{\sqrt{1+1+1}}=\sqrt{3}$



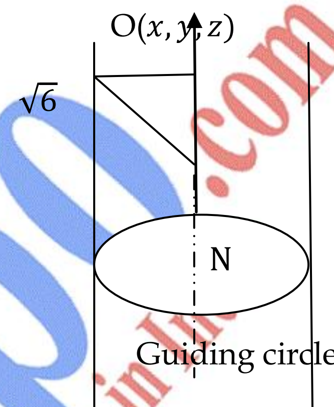
Therefore the radius of the circle = $\sqrt{r^2-p^2}=\sqrt{9-3}=\sqrt{6}$

Now any line perpendicular to the plane $x - y + z = 3$ will be the axix of the cylinder. Hence its equation are

$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-0}{1}$ (1)

Take a point P (x, y, z) on the cylinder. We draw PM perpendicular to (1) then PM= $\sqrt{6}$ now we join PO, from right Angle Δ OPM

$OP^2 = PM^2 + MO^2$



$\Rightarrow x^2+y^2+z^2=6, \left[\frac{1(x-0)-1(y-0)+1(z-0)}{\sqrt{1^2+1^2+1^2}}\right]^2$

$x^2+y^2+z^2+ xy + yz - zx - 9 = 0$

which is the equation of the required right circular cylinder

Q.25 Find the equation of a right circular cylinder whose guiding circle passes through the points (a,0,0), (0,b,0) and (0,0,c)

Sol. The equation of the circle through three given points are

$x^2+y^2+z^2- ax-by-cz=0$ (1)

and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (2)

The axis of the right circular cylinder will be perpendicular to the plane of the circle

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ Hence the direction rates of the axis are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$

Let P (α, β, γ) be any point on the surface of the cylinder, then the equations of the generator through this point are

$\frac{x-\alpha}{1/a} = \frac{y-\beta}{1/b} = \frac{z-\gamma}{1/c} = r$ **(say)** - (3)

The co-ordinates of any point on this line are $(\alpha + r/a, \beta + r/b,, \gamma + r/c)$

Since every generator intersects the guiding curve for some value of r the co-ordinates of this point will satisfy the eq[in] of the circle

$\therefore (\alpha + r/a)^2 \ +(\beta + r/b)^2 + (\gamma + r/c)^2$

$- a(\alpha + r/a)$-b $(\beta + r/b)$ -c $(\gamma + r/c)$=0 (4)

and $\frac{1}{a}\left(\alpha+\frac{r}{a}\right)+\frac{1}{b}\left(\beta+\frac{r}{b}\right)+\frac{1}{c}\left(\gamma+\frac{r}{c}\right)=1$ (5)

from Equation (4) we have

$r^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+2r\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-\frac{3}{2}\right)$

$(\alpha^2+\beta^2+\gamma^2-\alpha a-\beta b-\gamma c)=0$ - (6)

Also from Eq[in] (5) we have

$r=\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-1\right)/\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$ **(7)**

Now eliminate r from (6) and (7) we get

$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \qquad (\alpha^2+\beta^2+\gamma^2-\alpha a-\beta b-\gamma c)$

$=\left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=-1\right) \qquad \left(\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}-2\right)$

Hence the locus of P (α,β,γ) is

$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)(x^2+y^2+z^2-ax-by-cz)$

$=\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-2\right)$

which is the equation to the required right circular cylinder.

Q.26 A plane passes through a fixed point (a,b,c) and cut the axes in A,B,C. Show that the locus of the centre of the sphere OABC is

$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 2$

Sol. Let the co-ordinates of A,B,C be $(\alpha,0,0)$, $(0,\beta,0)$, $(0,0,\gamma)$ respectively, then the equation of the plane ABC is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ **(1)**

If (1) passes through (a,b,c) then

$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$ (2)

Now equation of any sphere through origin is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ (3)

if (3) passes through A (α,0,0), B (0, β,0), C(0,0, γ)

We have $2u + \alpha = 0 \Rightarrow u = -\alpha/2$

$2v + \beta = 0 \Rightarrow v = -\beta/2$

$2w + \gamma = 0 \Rightarrow w = -\gamma/2$

Hence the equation of the sphere OABC is $x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$ (4)

Let (f,g,h) be the centre of (4) then

$f=\frac{\alpha}{2},\ g=\frac{\beta}{2},\ h=\frac{\gamma}{2}$

Putting these values of α, β, γ in (2) and generalizing, we get the required locus as

$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

Unit-2

Central coracoids , Generating Lines

The general Equation of second degree (conicoid) :-

The surface represented by the general equation of second degree in x, y, z i.e. $ax^2 + by^2 + cz^2 + 2gzx + 2fyz + 2hxy + 2ux + 2vy + 2wz + d$ =0 is called a conicoid or quadric.

The standard equation of the central conicoid –

The surface represented by the equation $ax^2 + by^2 + cz^2$=⊥ (1) possesses the property that all chords of the surface which pass through the origin are bisected at the origin.

The conicoid represented by the equation (1) possesses a unique centre (origin) that is, the origin is the only point which possesses this property. This conicoid is therefore known as the central conicoid and the equation (1) is its standard equation.

The equation (1) represents three different surface depending upon the signs of the coefficients a,b,c. if a,b,c are all positive then the surface is an ellipsoid, if one of them is negative the surface is the hyperboloid of one sheet. In case all the three are negative the surface is imaginary.

Equation of the ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Equation of the hyperboloid of one sheet

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Equation of the hyperboloid of two sheets

$\frac{-x^2}{a^2} + \frac{y^2}{b^2} - \frac{-z^2}{c^2} = 1$

Tangent lines and tangent plane

Definition – A straight line which intersects a central conicoid in two coincident point is called a tangent line to the central conicoid at that point.

The locus of all tangent lines at a point on a central conicoid is called a tangent plane to the central conicoid at that point.

Equation of the tangent plane at a point → The equation of the tangent plane to the conicoid $ax^2 + by^2 + cy^2$=1 at a point (α, β, γ) is

$\boldsymbol{a}\alpha x + b\beta y + c\gamma z$=1

corollary- The equation to the tangent plane at (α, β, γ) to the ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$ =1 is $\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2}$ =1

Condition of tangency –

The condition that the plane $lx + my + nz$=P is a tangent plane to the conicoid $ax^2 + by^2 + cz^2$=1 is

$\frac{l^2}{a^2} + \frac{m^2}{b} + \frac{n^2}{c}$ =P²

Coordinates of tangent point

$(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp})$

Director sphere :-

Definition- The locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid is a sphere, concentric with the conicoid, called the Director sphere of the conicoid.

→ The locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid $ax^2 + by^2 + cz^2 = 1$ is

$x^2 + y^2 + z^2 = (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})$

corollary – The locus of the point of intersection of three mutually perpendicular tangent planes to the ellipsoid.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

is $\boldsymbol{x^2 + y^2 + z^2 = a^2 + b^2 + c^2}$

This sphere is called the director sphere of the ellipsoid

Pole and polar plane –

Definition :-if through a given point a any secant be drawn to meet the conicoid in P and Q then the locus of a point R on APQ such that AP, AR and AQ are in H.P. i.e.

AR = $\frac{2AP.AQ}{AP+AQ}$ is called the polar plane of A with respect to conicoid and the point A is said to be pole of this plane.

Equation of the polar line – The equation to the polar line of the given line

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ with respect to the conicoid $\boldsymbol{ax^2 + by^2 + cz^2 = 1}$ **is**

$\boldsymbol{a}\alpha x + b\beta y + c\gamma z = 1$

$\boldsymbol{a}lx + bmy + cnz = 0$

Enveloping cylinder

Definition- The locus of the tangent lines to a conicoid parallel to any given line is called the enveloping cylinder of the conicoid.

⇒ The Equation of the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$with its generators parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

is $(alx + bmy + cnz)^2 = (ax^2 + by^2 + cz^2 - 1)\ (al^2 + bm^2 + cn^2)$

Normal

Definition- The normal PN at any point P of a surface is the straight line passing through P and perpendicular to the tangent plane at P.

Equation of the normal

The Equation of the normal at the point P $(x_1, y_1, z_1)$of the ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ =1 is

$\frac{(x-x_1)}{\frac{x_1}{a^2}} = \frac{y-y_1}{\frac{y_1}{b^2}} = \frac{z-z_1}{\frac{z_1}{c^2}}$

Q.1 Show that the locus of a point, the sum of whose distances from points (a,0,0) and (-a,0,0) is constant (=2k), is the ellipsoid of revolution $\frac{x^2}{k^2}+\frac{y^2+z^2}{k^2-a^2}$ =1

Sol Here according the question

PA + PB = 2k

$\sqrt{(x+a)^2+y^2+z^2} + \sqrt{(x-a)^2+y^2+z^2}$ = 2k

$\sqrt{(x+a)^2+y^2+z^2}$= 2k – $\sqrt{(x-a)^2+y^2+z^2}$

$x^2+y^2+z^2+2ax+a^2 = 4k^2+x^2+y^2+z^2-2ax+a^2$-4k $\sqrt{(x-a)^2+y^2+z^2}$

$4ax-4k^2 = -4k(x-a)^2+y^2+z^2$

$(ax-k^2)^2 = k^2(x^2+y^2+z^2-2ax+a^2)$

$a^2x^2+k^4-2axk^2 = k^2x^2+k^2y^2+k^2z^2-2ak^2x+a^2k^2$

$x^2(a^2-k^2)-k^2y^2-k^2z^2 = a^2k^2-k^4$

$\frac{x^2(a^2-k^2)}{k^2\,(a^2-k^2)} - \frac{k^2(y^2+z^2)}{k^2\,(a^2-k^2)} = 1$

$\frac{x^2}{k^2}+\frac{y^2+z^2}{k^2-a^2}$ =1

Q.2 **Find the equation to the tangent plane at** (α, β, γ) to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2}$ =1

Sol. $\frac{x^2}{k^2} + \frac{y^2+z^2}{b^2}$ =(1) (given ellipsoid)

Equation (1) have point (α, β, γ) then

$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{b^2}$ =1 -(2)

The Equation of line passing through the point (α, β, γ) is

$$\frac{x-\alpha}{l} - \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (let)} \qquad - (3)$$

The point lie on this line $(\ell r + \alpha, mr + \beta, nr + \gamma)$ if these point are situated on central conicoid

$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2}$ =(1) then

$\frac{1}{a^2}(lr + \alpha)^2 + \frac{1}{b^2}(mr + \beta)^2 + \frac{1}{b^2}(nr + \gamma)^2 = 1$

$\Rightarrow r^2\left[\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{b^2}\right] + 2r\left[\frac{\ell\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma}{b^2}\right] + \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{b^2} - 1 = 0$

$\Rightarrow r^2\left[\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{b^2}\right] + 2r\left[\frac{\ell\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma}{b^2}\right] + 1 - 1 = 0$ (from eq. ())

$\Rightarrow r = 0,\ r\left[\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{b^2}\right] + 2\left[\frac{\ell\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma}{b^2}\right] = 0$

The other value of r will be equal to zero if

$\frac{\ell\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma}{b^2} = 0$

$r\left[\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{b^2}\right] = 0$

$\Rightarrow r = 0, \frac{\ell\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma}{b^2} = 0$ -(4)

New eliminate r from Eq. (3) (4)

$\frac{(x-\alpha)\alpha}{a^2} + \frac{(y-\beta)\beta}{b^2} + \frac{(z-\gamma)\gamma}{b^2} = 0$

$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{b^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{b^2}$

$\frac{\alpha x}{a^2} + \frac{\beta y + \gamma z}{b^2} = 1$

Which is the required equation of tangent plane?

Q.3 **Obtain the equation of the tangent plane at the point (α,β,γ) of $ax^2+by^2+cz^2$=1**

Sol. Let P (α,β,γ) be any point on central conicoid

therefore $a\alpha^2+b\beta^2+c\gamma^2$=1 - (1)

the line passing through the point (α,β,γ) is

$\frac{(x-\alpha)}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ (let) (2)

The point lie on this line $(\ell r+\alpha, mr+\beta, nr+\gamma)$

if this point are situated on $ax^2+by^2+cz^2$=1

then $a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2$=1

$\Rightarrow r^2(al^2+bm^2+cn^2)+2r(al\alpha+b\beta m+c\gamma n)+a\alpha^2+b\beta^2+c\gamma^2-1=0$

$\Rightarrow \quad r^2(al^2+bm^2+cn^2)+2r(al\alpha+b\beta m+c\gamma n)=0$

$r=0, r(al^2+bm^2+cn^2)+2(al\alpha+b\beta m+c\gamma n)$=0

the other value of r will be equal to zero

if $al\alpha+b\beta m+c\gamma n$=0

$r(al^2+bm^2+cn^2)$=0

$\Rightarrow r$ =0, $a\alpha l+b\beta m+c\gamma n$=0

Now eliminate the value of l. m. n from equation (2) and (3)

Which is the required equation of tangent plane

Q.4 **Find the equation to the two tangent planes of the conicoid $ax^2+by^2+cz^2$=1 which are parallel to the plane $lx+my+nz$=0**

Sol. Equation of conicoid $ax^2+by^2+cz^2$=1 --(1)

The equation of the plane, parallel to plane $lx+my+nz$ =0 is given by

$lx+my+nz$ =P --(2)

Equation (2) touch conicoid $ax^2+by^2+cz^2$=1 if

$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$=P^2

P= $\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$

$lx + my + nz = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$ – [from Eq. 2]

Q.5 Find the condition that the line $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ should be tangent line to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ at the point (α, β, γ) which lies on the ellipsoid.

Sol. The given Equation of the line

$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ (let) (1)

The point on the given line (1) $(lr + \alpha, mr + \beta, nr + \gamma$ If these point lie on $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ then $\frac{(lr+\alpha)^2}{a^2} + \frac{(mr+\beta)^2}{b^2} + \frac{(nr+\gamma)^2}{c^2} = 1$

$\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) r^2 + 2r \left(\frac{l\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma^2}{c^2}\right) = 0$

here one value of r is equal to zero. r=0 line (1) will be tangent line if other value of r also equal to zero for which necessary condition will be $\left(\frac{l\alpha}{a^2} + \frac{m\beta}{b^2} + \frac{n\gamma^2}{c^2}\right) = 0$

Q. 6 If the tangent plane to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ cuts off intercepts (α, β, γ) from the axes, show that $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} = 1$

Sol. The Equation to the tangent plane at (x, y, z) to the cllipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$\frac{1}{a^2} x x_1 + \frac{1}{b^2} y y_1 + \frac{1}{c^2} z z_1 = 1$

Equation (1) passes through point A (α, o, o), B (o, β, o), C (o, o, γ) therefore

$\frac{x_1 \alpha}{a^2} = 1$, $\Rightarrow x_1 = \frac{a^2}{\alpha}, y_1 = \frac{b^2}{\beta}, z_1 = \frac{c^2}{\gamma}$

The point (x_1, y_1, z_1) lie on ellipsoid

therefore $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$

Now substitute the value of x_1, y_1, z_1 in equation (2) we obtain

$\frac{a^4}{\alpha^2 a^2} + \frac{b^4}{\beta^2 b^2} + \frac{c^4}{\gamma^2 c^2} = 1$

$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} = 1$

Q.7 **Find the point of intersection of the line $\frac{x+5}{-3}$=y-4 =$\frac{z-11}{7}$ and the conicoid $12x^2 - 17y^2 + 7z^2 = 7$**

Sol. The given equation of line $\frac{x+5}{-3}$=$\frac{y-4}{1}$ =$\frac{z-11}{7}$ (rlet)

any point of given line $(-3r-5, r+4, 7r+11)$

The Equation of given conicoid $12x^2 - 17y^2 + 7z^2 = 7$

if this points lies on this conicoid then

$(12(-3r-5)^2 - 17(r+4)^2 \ \ + 7(7r+11)^2 = 7$

$\Rightarrow 12[9r^2 + 25 + 30r] - 17[r^2 + 16 + 8r] + 7[49r^2 + 121 + 154r)$

$\Rightarrow 108r^2 + 300 + 360r - 17r^2 - 272 - 136r + 343r^2 + 847 + 1078r - 7$=0

$\Rightarrow 434r^2 + 1302r + 828$ =0

$\Rightarrow 217r^2 + 615r + 434$=0

$r = \frac{-651 \pm \sqrt{47089}}{434} = \frac{-651 \pm 217}{434}$

$r = -1, -2$

Points (-2,3,4) and (1,2,-3)

Q.8 **Prove that the locus of the poles of the tangent planes of $ax^2 + by^2 + cz^2$=1 with respect to $\alpha x^2 + \beta y^2 + \gamma z^2$=1 is the following conicoid $\frac{\alpha^2 x^2}{a}$ + $\frac{\beta^2 y^2}{b}$+ $\frac{\gamma^2 z^2}{c}$ =1**

Sol. The equation of the tangent plane to the conicoid $ax^2 + by^2 + cz^2$=1 (1) is

$lx + my + n = P$ - (2)

then $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}$=P² - (3)

suppose $(\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1})$ be the pole of the plane (2) with respect to the conicoid (1) $\alpha x^2 + \beta y^2 + \gamma z^2$=1 (4)

then the equation of the polar will be

$\alpha x x_1 + \beta y y_1 + \gamma z z_1$=1 (5)

(2) and (5) are the equations of the same plane therefore they must be identical, on comparing the coefficients, we get

$\frac{\alpha x_1}{l} = \frac{\beta y_1}{m} = \frac{\gamma z_1}{n} = \frac{1}{P}$

$\therefore l = P\alpha x_1$, $m = P\beta y_1$, $n = P\gamma z_1$ - (6)

Eliminating l,m,n from (3) and (6) we get

$\frac{P^2\alpha^2 x_1^2}{a} + \frac{P^2\beta^2 y_1^2}{b} + \frac{P^2\gamma^2 z_1^2}{c} = P^2$

Hence the required locus of (x_1, y_1, z_1)

$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1$

Q.9 Find the length of the normal chord through P of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol. Let the co-ordinates of the point P (α, β, γ) then the Equations of the normal at P in terms of actual directions cosines are

$\frac{x-\alpha}{P\alpha/a^2} = \frac{y-\beta}{P\beta/b^2} = \frac{z-\gamma}{P\gamma/c^2}$ = r (say)

where P = $\frac{1}{\sqrt{\alpha^2/a^4+\beta^2/b^4+\gamma^2/c^4}}$ is the perpendicular distance of the origin from the tangent at P

Now the Coordinates of any point say

Q on the normal at P are

$$\left(\alpha + \frac{P\alpha r}{a^2}, \beta + \frac{P\beta r}{b^2}, \gamma + \frac{P\gamma r}{c^2}\right)$$

if Q also lies on the ellipsoid then

$$\frac{\left(\alpha+\frac{P\alpha r}{a^2}\right)^2}{a^2} + \frac{\left(\beta+\frac{P\beta r}{b^2}\right)^2}{b^2} + \frac{\left(\gamma+\frac{P\gamma r}{c^2}\right)^2}{c^2} = 1$$

$$r^2P^2\left(\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right) + 2rP\left(\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right) + \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right) = 1$$

$$r^2P^2\left(\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right) + 2rP\left(\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right) = 0$$

$\therefore$ P lies on the ellipsoid)

$r = 0$, and r= $\frac{-2\Sigma(\alpha^2/a^4)}{P\Sigma(\alpha^2/a^6)}$

$= \frac{-2lP^2}{P\Sigma(\alpha^2/a^6)} = \frac{-2}{P^3\Sigma(\alpha^2/a^6)}$

<u>Hence the length of the normal</u> $= \frac{2}{P^3\Sigma(\alpha^2/a^6)}$

Q.10 Prove that the feet of the six normal's drawn to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$=1 from any point (α,β,γ) lies on the curve of intersection of ellipsoid and the cone $\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$

Sol. Equation of the normal at (f,g,h) are

$\frac{x-f}{f/a^2}=\frac{y-g}{g/b^2}=\frac{z-h}{h/c^2}$ - (1)

if (1) passes through (α,β,γ) then

$\frac{\alpha-f}{f/a^2}=\frac{\beta-g}{g/b^2}=\frac{\gamma-h}{h/c^2}$= K (say)

then the co-ordinates of six feet of normals from (α,β,γ) are given by

f = $\frac{a^2\alpha}{a^2+K}$, g = $\frac{b^2\beta}{b^2+K}$, h=$\frac{c^2\gamma}{c^2+K}$

k= $\frac{a^2\alpha}{f}-a^2, k=\frac{b^2\beta}{g}-b^2, k=\frac{c^2\gamma}{h}-c^2$

Now multiplying these values of K by $(b^2-C^2),(C^2-a^2)$ and (a^2-b^2) respectively and adding we have

$$0=\frac{a^2\alpha}{f}(b^2-C^2)+\frac{b^2\beta}{g}(c^2-a^2)+\frac{c^2\gamma}{h}(a^2-b^2)$$

∴ The locus of (f,g,h) the feet of the normals is

$$\frac{a^2\alpha(b^2-c^2)}{x}+\frac{b^2\beta(c^2-a^2)}{y}+\frac{c^2\gamma(a^2-b^2)}{z}=0$$

This is a homogenous Equation therefore represents a cone. Also the feet of the normals lie on the ellipsoid.

Hence the feet of the six normals lie on the curve of intersection of the ellipsoid and the cone (2)

Q. 11 The section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ where vertex if P, by the plane Z=0 is a rectangular hyperbola prove that the locus of P is $\frac{x^2+Y^2}{a^2+b^2}+\frac{z^2}{c^2}$=1

Sol. Suppose co-ordinate of vertex of given enveloping cone of the ellipsoid is (α,β,γ) then the Equation of enveloping cone will be

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$$

$= \left(\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}-1\right)^2$........... **(1)**

$[SS_1=T^2]$

Now z=0

$\therefore \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\ \left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right) = \left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)$- (2)

according to the question (2) represent a rectangular hyperbola then coefficient of x^2 + coefficient of y^2=0

$\frac{1}{a^2}\left(\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)=0$

$\frac{\alpha^2+\beta^2}{a^2b^2}+\frac{\gamma^2}{c^2}\ \left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{1}{a^2}+\frac{1}{b^2}$

$\frac{\alpha^2+\beta^2}{a^2+b^2}+\frac{\gamma^2}{c^2}=1$

The locus of (α,β,γ) is $\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$

Q. 12 The section of an enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane z=0 is a parabola show that the locus of the vertices of the cone is z= $\pm C$

Sol. Let P (α,β,γ) be a luminous point then the equation of the enveloping cone with vertex (α,β,γ) is $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$

$\left(\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}-1\right)^2$ - (1)

$[SS_1=T^2]$

then the section of (1) by Z=0 plane is

$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\ \left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2$

$\Rightarrow \frac{x^2}{a^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)+\frac{y^2}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)-\frac{2\alpha\beta}{a^2b^2}xy=0$----(2)

according to question its represent a parabola

$\therefore \frac{1}{a^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)=\frac{\alpha^2\beta^2}{a^4b^4}$ [From ab = n^2]

$\Rightarrow \left(\frac{\gamma^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=0$

Therefore the required locus of (α,β,γ) is

$\Rightarrow \left(\frac{z^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$

$\frac{z^2}{c^2}-1=0;\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$

here second equation represent a ellipsoid therefore point (α,β,γ) will not lie on this ellipsoid therefore. The locus of vertex (α,β,γ) is

$\frac{z^2}{c^2}=1 \Rightarrow z=\pm C$

Q. 13 Find the locus of points from which three mutually perpendicular tangents can be drawn to the surface $ax^2+by^2+cz^2=1$

Sol. Let P (α,β,γ) be a point from which three mutually perpendicular tangents can be drawn to the surface $ax^2+by^2+cz^2=1$, then the enveloping cone with P as its vertex will have three mutually perpendicular generators.

The equation of the enveloping cone is **[$SS_1=T^2$]**

$(ax^2+by^2+cz^2-1)\ (a\alpha^2+b\beta^2+c\gamma^2-1)=(a\alpha x\ \ +b\beta y^2+c\gamma z\ \ -1)^2$-1 will have three mutually perpendicular generators if the sum of the coefficients of x^2, y^2 and z^2 is equal to zero i.e.

$a(b\beta^2+c\gamma^2-1)+b(a\alpha^2+c\gamma^2-1)+c(a\alpha^2+b\beta^2-1)=0$

$a(b+c)\alpha^2+b(c+a)\beta^2+c(a+b)\gamma^2=a+b+c$

Hence the locus of $(\alpha,\beta,\gamma)$is

$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c$

Q. 14 Find the locus of centres of sections of $ax^2+by^2+cz^2=1$ which touch $\alpha x^2+\beta y^2+\gamma z^2=1$

Sol. Let (f,g,h) be the centre of a section of the conicoid $ax^2+by^2+cz^2=1$, which touches the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$ ------(1)

Now the equation of the section is

$af(x-f)+bg\ (y-g)+ch(z-h)=0$

$\Rightarrow axf+byg+czh=af^2+bg^2+ch^2$ --(2)

(1) touches (1), therefore

$\frac{(af)^2}{\alpha}+\frac{(bg)^2}{\beta}+\frac{(ch)^2}{\gamma}=(af^2+bg^2+ch^2)^2$

Hence the locus of (f,g,h) is

$\frac{a^2x^2}{\alpha}+\frac{b^2y^2}{\beta}+\frac{c^2z^2}{\gamma}=(ax^2+by^2+cz^2)^2$

Q. 15 **Find the equation to the two planes which contain the line given by $7x+10y-30=0$, $5y-3z=0$ and touches the ellipsoid $7x^2+5y^2+3z^2=60$**

Sol. Equation of any plane through given line is

$7x+10y-30+\lambda(5y-3z)=0$

$\Rightarrow 7x+(10+5\lambda)y-3\lambda z-30=0$ - (1)

Let this touches the given ellipsoid at the point (α,β,γ)

The equation of the given Ellipsoid is $7x^2 + 5y^2 + 3z^2$=60

$\Rightarrow \frac{7}{60}x^2 + \frac{5}{60}y^2 + \frac{3}{60}z^2 = 1$ (2)

$\therefore$ The equation of the tangent to (2) at (α, β, γ) is

$\frac{7}{60}\alpha x + \frac{1}{12}\beta y + \frac{1}{20}\gamma z = 1$ (3)

(1)and (3) represents the same plane, then there they must be identical , on comparing the coefficient, we get

$\frac{\alpha}{60} = \frac{\beta}{12(10+5\lambda)} = \frac{\gamma}{20(-3\lambda)} = \frac{1}{30}$

$\Rightarrow \alpha=2,\ \beta = \frac{2(10+5\lambda)}{5} = \ \gamma = -2\lambda$

point (α, β, γ) lies on (3) therefore

$7\alpha^2 + 5\beta^2 + 3\gamma^2 = 60$

$\Rightarrow 28 + \frac{20(10+5\lambda)^2}{25} + 3(-2\lambda)^2 = 60$

$\Rightarrow 2\lambda^2 + 5\lambda + 3 = 0$

$\Rightarrow (\lambda+1)(2\lambda+3) = 0 \qquad \therefore \lambda = -1, -3/2$

On putting the values of λ in (1) we get the required equations of tangent planes as

$7x + 5y + 3z - 30 = 0$

and $14x + 5y + 9z - 60 = 0$

Q. 16 The tangent planes to the ellipsoid $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right)$ meets the coordinate axes in points A,B and C. Prove that the centroid of the triangle ABC lies on the locus $\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 9$

Sol. Equation of any tangent plane to the ellipsoid is

$lx + my + nz = \sqrt{a^2l^2 + b^2m^2 + c^2n^2}$ -- (1)

1. meets the coordinate axis i.e. x, y, and Z-axis at the points A,B and C respectively then the co-ordinates of A are

$\left(\frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{l}, 0,0\right)$ etc

Let (f,g,h) be the coordinates of the centroid of the triangle ABC then

F = $\frac{1}{3}\left[\frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{l} + 0 + 0\right]$

or $\frac{a^2}{f^2} = \frac{9a^2l^2}{a^2l^2+b^2m^2+c^2n^2}$

similarly $\frac{b^2}{g^2} = \frac{9b^2m^2}{a^2l^2+b^2m^2+c^2n^2}$

and $\frac{c^2}{n^2}=\frac{9c^2n^2}{a^2l^2+b^2m^2+c^2n^2}$

for finding the locus of (f,g,h) we have to eliminate l,m,n by adding the above relations and generalize (f,g,h) we have the required locus as

$\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}=9$

Q. 17 **Find the equations to the generating lines of the hyperboloid $yz+2zx+3xy+6=0$ which pass through the point (-1,0,3)**

Sol. The equation of the given hyperboloid is

$\boldsymbol{yz+2zx+3xy+6=0}$ (1)

This can be written as

$(y+2)(z+3)+(2z+3y)(x-1)=0$

or $(y+2)(z+3)-(2z+3y)(1-x)=0$ (2)

or $\frac{y+2}{2z+3y}=\frac{1-x}{z+3}=\lambda$ (say)

and $\frac{y+2}{1-x}=\frac{2z+3y}{z+3}=\mu$ (say)

or $y+2=\lambda(2z+3y)$, $z+3=\frac{1}{\lambda}(1-x)$ (3)

$y+2=\mu(1-x)$, $z+3=\frac{1}{\mu}(2z+3y)$ (4)

above are λ and μ - generators system of (1) if (3) and (4) pass through (-1,0,3) then 2=λ(b) or $\lambda=\frac{1}{3}$ and 2 = μ (1+1) Or μ =1

Putting the value of λ in (3) , we get z=3, $3x+z=0 \Rightarrow x=-1, z=3$ -- (5)

Again putting the value of H in (4), we get

$x+y+1=0$, $z+3y=3$, $\Rightarrow \frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$ (6)

(5) and (6) gives the Equations of generating lines of the given hyperboloid which pass through the point (-1,0,3)

Q. 18 Find the equations to the generating lines of the hyperboloid $\left(\frac{x^2}{4}+\frac{y^2}{9}+\frac{-z^2}{16}=1\right)$ which pass through the point (2,3,-4)

Sol. Let the equations of the generator through (2,3-4) are

$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r$ (say) - (1)

Co-ordinates of any point on (1) are

$(lr+2, mr+3, nr-4)$

This point will lie on the given hyperboloid if

$\frac{(lr+2)^2}{4}+\frac{(mr+3)^2}{9}-\frac{(nr-4)^2}{16}=1$

or if $r^2\left(\frac{l^2}{4}+\frac{m^2}{9}+\frac{-n^2}{16}\right)+2r\left(\frac{l}{2}+\frac{m}{3}+\frac{n}{4}\right)=0$ (2)

Since the line (1) is the generator of the given hyperboloid, therefore (2) will be true for all values of r i.e. an identity, hence

$\frac{l^2}{4}+\frac{m^2}{9}+\frac{-n^2}{16}=0$ (3)

and $\frac{l}{2}+\frac{m}{3}+\frac{n}{4}=0$ (4)

Eliminating n from (3) by the help of (4) we get

$\frac{l^2}{4}+\frac{m^2}{3}-\left(\frac{l}{2}+\frac{m}{3}\right)^2=0$

$\Rightarrow lm=0 \Rightarrow l=0$, or m=0

if l=0 then

$\frac{m^2}{9}-\frac{n^2}{16}=0$ and $\frac{m}{3}+\frac{n}{4}=0$

$\Rightarrow \frac{m}{3}=\frac{n}{-4}$ i.e. $\frac{l}{o}=\frac{m}{3}=\frac{n}{-4}$

if m=0 then

$\frac{l^2}{4}-\frac{n^2}{16}=0$ and $\frac{l}{2}+\frac{n}{4}=0$

$\Rightarrow \frac{l}{1}=\frac{n}{-2}$ i.e. $\frac{l}{1}=\frac{m}{0}=\frac{n}{2}$

Hence the required generators through the point (2,3-4) are

$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4}$

and $\frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}$

Q. 19 Find the equations to the generator of the hyperboloid $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{-z^2}{c^2}=1\right)$ which pass through the point (a cos $\alpha, b\ Sin\ \alpha$ 0)

Sol. Any point on the principal elliptic section of the hyperboloid is (a cos $\alpha, b\ Sin\ \alpha$ 0)
Let the equations of the generator through the point (a cos $\alpha, b\ Sin\ \alpha, 0$) be

$$\frac{x-a\cos\alpha}{l}=\frac{y-b\sin\alpha}{m}=\frac{z-0}{n}=\text{r(say)} \quad \text{------ (1)}$$

Coordinates of any point on (1) are

$(lr+a\ cos\boldsymbol{\alpha}, \boldsymbol{mr}+\boldsymbol{bsin\alpha}, \boldsymbol{nr})$

As (1) is the equations of generator of the given hyperboloid therefore every point of (1) will lie on the hyperboloid.

therefore $\frac{1}{a^2}(lr+\text{a}\cos\alpha)^2+\frac{1}{b^2}(mr+\text{b}\sin\alpha)^2-\frac{1}{c^2}n^2r^2=1$

or $\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)r^2+2\left(\frac{l\cos\alpha}{a}+\frac{m\sin\alpha}{b}\right)r=0$ ----------- **(2)**

now (2) must be an identity as it is satisfied for all values of r, therefore

$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0$ ----------- **(3)**

and $\frac{l\cos\alpha}{a}+\frac{m\sin\alpha}{b}=0$ ----------**(4)**

now from (4) we get $\frac{l\cos\alpha}{a}=\frac{-m\sin\alpha}{b}$

$\Rightarrow \pm\frac{l/a}{\sin\alpha}=\frac{m/b}{-\cos\alpha}=\pm\frac{\sqrt{l^2/a^2+m^2/b^2}}{\sqrt{Sin^2\alpha+Cos^2\alpha}}$

$\Rightarrow \frac{l}{a\sin\alpha}=\frac{m}{-b\cos\alpha}=\frac{\pm n/c}{1}$ from (3)

∴ The equation of the required generators are as follows from (1) and (5)

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\pm C}$$

Properties of generating lines of hyperboloid of one sheet

Property -1 No two generators of the same system of a hyperboloid of one sheet interest.

Sol. Let the two generators of the λ - system for two different values of λ say $λ_1$ *and* $λ_2$ *be*

$\frac{x}{a}+\frac{z}{c}=\lambda_1\ (1+\frac{y}{b})$, $\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda_1}(1-\frac{y}{b})$----------(1)

And $\frac{x}{a}+\frac{z}{c}=\lambda 2\ (1+\frac{y}{b})$, $\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda_2}(1-\frac{y}{b})$----------(2)

Subtracting first equations of (1) and (2), we get

$(\lambda_1-\lambda_2)\ (1+\frac{y}{b})=0 \Rightarrow y=-b$ because $\lambda_1 \neq \lambda 2$

similarly subtracting second equations of (1) and (2) we obtain

$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda 2}\right)(1-\frac{y}{b})=0 \Rightarrow y=b$ because $\lambda_1 \neq \lambda 2$

from above we observe that the equations (1) and (2) representing two generators of the same system are inconsistent and therefore its follow that the two generators of the same system do not intersect.

Property -2 Any generators of the λ system intersects any generator of the μ -system of hyperboloid of one sheet.

Solution Let the two generators one of each system be

$\frac{x}{a}+\frac{z}{c}=\lambda\ (1+\frac{y}{b})$, $\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}(1-\frac{y}{b})$ ---------- (1)

$\frac{x}{a}-\frac{z}{c}=\mu\ (1+\frac{y}{b})$, $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}(1-\frac{y}{b})$ ---------- (2)

they will intersect only if one point say (x, y, z) satisfy all the four equation of (1) and (2)

For finding the point of intersection of these generators, from Equations (1) and (2), we have $\lambda\left(1+\frac{y}{b}\right)=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$

or $\frac{y}{b}=\frac{1-\lambda\mu}{1+\lambda\mu}$ ---------(3)

$\therefore \frac{x}{a}+\frac{z}{c}=\left(\frac{2\lambda}{1+\lambda\mu}\right)$ and $\frac{x}{a}-\frac{z}{c}=\frac{2\mu}{1+\lambda\mu}$

Solving these, we obtain $\frac{x}{a}=\left(\frac{\lambda+\mu}{1+\lambda\mu}\right)$ and $\frac{z}{c}=\frac{\lambda-\mu}{1+\lambda\mu}$ ---------(4)

Hence both the generators intersect and the coordinates of the point of their

$\left[\frac{a(\lambda+\mu)}{1+\lambda\mu}), \frac{b(1-\lambda\mu)}{1+\lambda\mu}, \frac{c(\lambda-\mu)}{1+\lambda\mu}\right]$

Property -3 The plane through two intersecting generators is the tangent plane of the hyperboloid of one sheet at their common point.

Solution Equation of any plane through λ-generator is

$$\left\{\left(\frac{x}{a}+\frac{z}{c}\right)-\lambda\left(1+\frac{y}{b}\right)\right\}+K\left\{\left(\frac{x}{a}-\frac{z}{c}\right)\right\}-\frac{1}{\lambda}\left\{1-\frac{y}{b}\right\}=0 \quad \text{---------- (1)}$$

Equation of any plane through H-generator is

$\frac{x}{a} = \left(\frac{\lambda+\mu}{1+\lambda\mu}\right)$ and $\frac{z}{c} = \frac{\lambda-\mu}{1+\lambda\mu}$

$$\left\{\left(\frac{x}{a}-\frac{z}{c}\right)-\mu\left(1+\frac{y}{b}\right)\right\}+K^1\left\{\left(\frac{x}{a}+\frac{z}{c}\right)\right\}-\frac{1}{\mu}\left\{1-\frac{y}{b}\right\}=0 \quad \text{---------------(2)}$$

where K and K′ are parameters.

Equations (1) and (2) becomes identical if we take $K = \frac{1}{K'} = \frac{\lambda}{\mu}$ Thus the two generator one of each system are co planer and as such they intersect and the equation of the plane containing them is

$$\frac{x}{a}(\lambda+\mu)+\frac{y}{b}(1-\lambda\mu)-\frac{z}{c}(\lambda-\mu) = 1+\lambda\mu \quad \text{---------------------(3)}$$

$$\Rightarrow \frac{x}{a} = \left(\frac{\lambda+\mu}{1+\lambda\mu}\right)+\frac{y}{b}\,\frac{1-\lambda\mu}{1+\lambda\mu}-\frac{z}{c}\left(\frac{\lambda-\mu}{1+\lambda\mu}\right) = 1$$

But this is the equation of the tangent plane to the hyperboloid of one sheet at the point of intersection of two generators one of each system i.e.

$$\left[\frac{a(\lambda+\mu)}{1+\lambda\mu}, \frac{b(1-\lambda\mu)}{1+\lambda\mu}, \frac{c(\lambda-\mu)}{1+\lambda\mu}\right]$$

Hence a tangent plane at a point of hyperboloid of one sheet meets the hyperboloid of one sheet in two generators through the point of contact.

Property **The locus of the point of intersection of perpendicular generators is the curve of intersection of the hyperboloid and the director sphere.**

Ans. Let the equation of hyperboloid of one sheet be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1 \quad \text{------------- (1)}$$

The equation of λ-system generators can be written as

$$\frac{x}{a}+\frac{z}{c} = \lambda\left(1+\frac{y}{b}\right) \Rightarrow \frac{x}{a}-\lambda\left(\frac{y}{b}\right)+\frac{z}{c} = \lambda$$

$$\frac{x}{a}-\frac{z}{c} = \frac{1}{\lambda}\left(1-\frac{y}{b}\right) \Rightarrow \frac{x}{a}+\frac{y}{\lambda b}-\frac{z}{c} = \frac{1}{\lambda} \quad \text{------------ (2)}$$

if l, m,n be the direction ratios of the generator then

$$\frac{l/a}{\lambda^2-1} = \frac{m/b}{2\lambda} = \frac{n/c}{\lambda^2+1} = \quad \text{----------- (3)}$$

Similarly the equation of μ-generator can be written as

$\frac{x}{a} - \mu\frac{y}{b} - \frac{z}{c} = \mu, \quad \frac{x}{a} + \frac{y}{\mu b} + \frac{z}{c} = \frac{1}{\mu}$ ------------ (4)

if l', m', n' be direction ratios of μ generators then

$\frac{l^1/a}{\mu^2-1} = \frac{m^1/b}{2\mu} = \frac{n^1/c}{\mu^2+1}$ -------------- (5)

if the two generators are perpendicular to each other then

$ll' + mm' + nn' = 0$

$\Rightarrow a^2(\lambda^2 - 1)(\mu^2 - 1) + 4b^2 \lambda \mu - C^2(\lambda^2 + 1)(\mu^2 + 1) = 0$

(by using (2) and (4))

$\Rightarrow a^2(\lambda + \mu)^2 + b^2(1 - \lambda\mu)^2 + C^2(\lambda - \mu)^2$

$= (\boldsymbol{a}^2 + \boldsymbol{b}^2 - \boldsymbol{c}^2)\ (\mathbf{1} + \lambda\mu)^2$

$\left(a\frac{\lambda+\mu}{1+\lambda\mu}\right)^2 = \left(b\frac{1-\lambda\mu}{1+\lambda\mu}\right)^2 + \left(c\ \frac{\lambda-\mu}{1+\lambda\mu}\right)^2 = a^2 + b^2 - c^2$

which is the condition that the point of intersection of two generators each one of them is a member of the λ-system, H-system respectively lie on a sphere $(\boldsymbol{x}^2 + \boldsymbol{y}^2 - \boldsymbol{z}^2) = \boldsymbol{a}^2 + \boldsymbol{b}^2 - \boldsymbol{c}^2$ known as director sphere.

Hence the locus of the point of intersection of perpendicular generator is the curve of intersection of director sphere and hyperboloid of one sheet.

Q. 20 The equations of the generating lines through a point $\theta, \emptyset$ of a hyperboloid of one sheet are $\left[\frac{\mathbf{x - a\cos\theta\sec\emptyset}}{\mathbf{\sin(\theta \pm \emptyset)}} = \frac{\mathbf{y - b\sin\theta\sec\emptyset}}{\mathbf{-b\cos(\theta \pm \emptyset)}} = \frac{\mathbf{z - c\tan\emptyset}}{\mathbf{\pm c}}\right]$

Sol. The direction ratios of λ-generator through a point 'P' α on the hyperboloid are a sin α, -b cos α, -c and the direction ratios of the μ-generators through the point $'\theta'$, $'\beta'$ on the hyperboloid are a sin β, - b cos β, c.

If these generators intersect at the point R " $\theta, \emptyset$" then $\alpha = \theta - \emptyset$ and $\beta = \theta + \emptyset$ therefore the direction ratio of λ-generator and μ- generator through the point R " $\theta, \emptyset$" will be

a sin $(\theta - \emptyset)$, -b cos $(\theta - \emptyset), -c$

a sin $(\theta + \emptyset)$, -b cos $(\theta + \emptyset)$, C respectively

Hence the equations of the λ and μ- generators through the point R (a cos θ, sec ∅, b sin θ sec ∅, c tan ∅ are

$$\frac{\mathbf{x-a\cos\theta\sec\emptyset}}{\mathbf{a\sin(\theta\pm\emptyset)}} = \frac{\mathbf{y-b\sin\theta\sec\emptyset}}{\mathbf{-b\cos(\theta\pm\emptyset)}} = \frac{\mathbf{z-c\tan\emptyset}}{\mathbf{\pm c}}$$