Biyani's Think Tank

Concept based notes

Numerical Methods

B.Sc-II

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this endeavour. They played an active role in coordinating the various stages of this endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Syllabus

Unit 1: Differences, Relation between differences and derivatives. Differences of a polynomial. Newton's formulae for forward and backward interpolation. Divided differences. Newton's divided difference, Interpolation formula. Lagrange's interpolation formula.

Unit 2: Central differences. Gauss's, Stirling's and Bessel's interpolation formulae. Numerical Differentiation. Derivatives from interpolation formulae. Numerical integration, Newton-Cote's formula, Trapezoidal rule, Simpson's one-third, Simpson's three-eighth and Gauss's quadrature formulae.

Unit 3: Numerical solution of algebraic and transcendental equations. Bisection method, Regula-Falsi method, Method of iteration, Newton-Rapshon method. Gauss elimination and Iterative methods (Jacobi and Gauss Seidal) for solving system of linear algebraic simultaneous equations. Solutions of ordinary differential equations of first order with initial and boundary conditions using Picard's and modified Euler's method.

Unit 4: Scalar point function. Vector point function. Differentiation and integration of vector point functions. Directional derivative. Differential operators—Gradient, Divergence and Curl. Theorems of Gauss, Green, Stokes (without proof) and problems based on these theorems.

Unit 1

Interpolation

Forward Difference

Q.1. **Construct a forward difference table for the following given data.**

x	**3.60**	**3.65**	**3.70**	**3.75**
y	**36.598**	**38.475**	**40.447**	**42.521**

Ans.:

x	**y**	**Δy**	**$\Delta^2 y$**	**$\Delta^3 y$**
3.60	36.598	1.877	0.095	0.007
3.65	38.475	1.972	0.102	
3.70	40.447	2.074		
3.75	42.521			

□

Backward Difference

Q.1. Construct a backward difference table form the following data :

sin 30° = 0.5000, sin 35° = 0.5736, sin 40° = 0.6428, sin 45° = 0.7071

Assuming third difference to be constant find the value of sin 25°.

Ans.:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
25	?			
		∇y_{30} =?		
30	0.5000		$\nabla^2 y_{35}$ = ?	
		0.0736		$\nabla^3 y_{40}$ = ?
35	0.5736		-0.0044	
		0.0692		-0.0005
40	0.6428		-0.0049	
		0.0643		
45	0.7071			

Since we know that $\nabla^3 y$ should be constant so

$\nabla^3 y_{40} = -0.0005$

$\Rightarrow \nabla^2 y_{40} - \nabla^2 y_{35} = -0.0005$

$\Rightarrow -0.0044 - \nabla^2 y_{35} = -0.0005$

$\nabla^2 y_{35} = +0.0005 - 0.0044$

$= -0.0039$

Again $\nabla^2 y_{35} = -0.0039$

$\nabla y_{35} - \nabla y_{30} = -0.0039$

$\Rightarrow 0.0736 - \nabla y_{30} = -0.0039$

$\nabla y_{30} = 0.0039 + 0.0736$

$= 0.0775$

Again $\nabla y_{30} = 0.0775$

$y_{30} - y_{25} = 0.0775$

$\Rightarrow 0.5000 - y_{25} = 0.0775$

$y_{25} = 0.5000 - 0.0775$

$= 0.4225$

Hence $\sin 25° = 0.4225$

□ □ □

Newton Gregory Formula for Forward Interpolation

Q.1. Use Newton formula for interpolation to find the net premium at the age 25 from the table given below :

Age	**20**	**24**	**28**	**32**
Annual net premium	**0.01427**	**0.01581**	**0.01772**	**0.01996**

Ans.:

Age (x)	**f(x)**	**Δf(x)**	**$\Delta^2 f$(x)**	**$\Delta^3 f$(x)**
20	0.01427			
		0.00154		
24	0.01581		0.00037	
		0.00191		-0.00004
28	0.01772		0.00033	
		0.00224		
32	0.01996			

Here a = 20 , h = 4 and k=2

$$u = \frac{x - x_k}{h} = 0.25$$

Using following Newton's Gregory forward interpolation formula :

$$y(x) = y_k + \Delta y_k u + \frac{\Delta^2 y_k}{2!} u(u-1) + ... + \frac{\Delta^{n-k} y_k}{(n-k)!} u(u-1).....(u-((n-k)-1)) \quad \Rightarrow$$

$$f(25) = 0.01581 + 0.00191(0.25) + \frac{0.00033}{2x1}(.75(0.25))$$

$$= 0.0162543$$

Q.2. From the following table find the number of students who obtained less than 45 marks :

Marks	**No. of Students**
30 – 40	**31**
40 – 50	**42**
50 – 60	**51**
60 – 70	**35**
70 – 80	**31**

Ans.:

Marks (x)	**No. of Students f(x)**	**Δf(x)**	**$\Delta^2 f$(x)**	**$\Delta^3 f$(x)**	**$\Delta^4 f$(x)**
Less than 40	31				
		42			
Less than 50	73		9		
		51		-25	
Less than 60	124		-16		37
		35		12	
Less than 70	159		-4		
		31			
Less than 80	190				

Here a = 40 , h = 10 and k=1

$$u = \frac{x - x_k}{h} = 0.5$$

using following forward interpolation formula :

$$y(x) = y_k + \Delta y_k u + \frac{\Delta^2 y_k}{2!} u(u-1) + ... + \frac{\Delta^{n-k} y_k}{(n-k)!} u(u-1) .....(u - ((n-k)-1))$$

$$= 31 + 42(0.5) + \frac{9}{2} 0.5(-0.5) + \frac{-25}{6} 0.5(-0.5)(-1.5) + \frac{37}{24} 0.5(-0.5)(-1.5)(-2.5)$$

= 47.8672 = 48 (approximately)

Hence the no. of students who obtained less than 45 marks are 48.

Q.3. Find the cubic polynomial which takes the following values

x	**0**	**1**	**2**	**3**
f(x)	**1**	**0**	**1**	**10**

Find f(4)

Ans.: Here we know that a = 0, h = 1 then form Newton's Gregory forward interpolation formula.

$P_n(x) = f(0) + {}^{x}c_1\,\Delta f(0) + {}^{x}c_2\,\Delta^2 f(0) + \_\ \_\ \_\ {}^{x}c_n\,\Delta^n f(0)$ _ _ _ _ (1)

(x)	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		$f(4) - 27 = 6$
3	10		$f(4) - 19$	(it should be constant)
		$f(4) - 10$		
4	$f(4)$			

Substituting the values in equation (1) from above table :

$$P_3(x) = 1 + x(-1) + \frac{x(x-1)}{1 \times 2}(2) + \frac{x(x-1)(x-2)}{1 \times 2 \times 3}(6)$$

$$P_3(x) = 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$= x^3 - 2x^2 + 1$$

Hence the required polynomial of degree three is

$$x^3 - 2x^2 + 1$$

Again $f(4) - 27 = 6$

$$\Rightarrow f(4) = 33$$

□ □ □

Newton's Formula for Backward Interpolation

Q.1. The population of a town in decennial census was as given below :

Year	**1891**	**1901**	**1911**	**1921**	**1931**
Population (in thousands)	**46**	**66**	**81**	**93**	**101**

Estimate the population for the year 1925.

Ans.:

Year (x)	**Population (in thousand) f(x)**	**∇f(x)**	**$\nabla^2 f$(x)**	**$\nabla^3 f$(x)**	**$\nabla^4 f$(x)**
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here x = 1925, h = 10 , k=5

$u = \frac{x - x_k}{h}$ =1925 – 1931/10 = -0.6

Now using Newton's Backward interpolation formula :

$$y(x) = y_k + \nabla y_k u + \frac{\nabla^2 y_k}{2!} u \;\; u+1 \;\; + ... + \frac{\nabla^{k-1} y_k}{k-1\;!} u \;\; u+1 \;\; ......(u + ((k-1)-1))$$

Put all the values we get

= 96.6352 thousand (approximately)

Divided Difference Interpolation

Q1. **Construct a divided difference table from the following data :**

x	**1**	**2**	**4**	**7**	**12**
$f(x)$	**22**	**30**	**82**	**106**	**216**

Ans.:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1} = 8$			
2	30		$\frac{26-8}{4-1} = 6$		
		$\frac{82-30}{4-2} = 26$		$\frac{(-3.6-6)}{7-1} = -1.6$	
4	82		$\frac{8-26}{7-2} = -3.6$		$\frac{0.535-(-1.6)}{12-1}$ $= 0.194$
		$\frac{106-82}{7-4} = 8$		$\frac{1.75-(-3.6)}{12-2} =$ 0.535	

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
7	106		$\frac{22-8}{12-4}$ = 1.75		
		$\frac{216-106}{12-7}$ = 22			
12	216				

Q.2. By means of Newton's divided difference formula find the value of $f(2)$, $f(8)$ and $f(15)$ from the following table :

x	**4**	**5**	**7**	**10**	**11**	**13**
$f(x)$	**48**	**100**	**294**	**900**	**1210**	**2028**

Ans.: Newton's divided difference formula for 4, 5, 7, 10, 11, 13 is :

$$f(x) = f(4) + (x-4)\underset{5}{\Delta} f(x) + (x-4)(x-5)\underset{5,7}{\Delta^2} f(4) + (x-4)(x-5)(x-7)\underset{5,7,10}{\Delta^3} f(4)$$

$$+ (x-4)(x-5)(x-7)(x-10)\underset{5,7,10,11}{\Delta^4} f(4) + \_\_\_\_\_\_\_ \quad \_\_\_ (1)$$

So constructing the following divided difference table :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100		$\frac{97-52}{7-4} = 15$		
		$\frac{294-100}{7-4} = 97$		$\frac{21-15}{10-4} = 1$	

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
7	294		$\frac{202-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

Substituting the values from above table in equation (1)

$$f(x) = 48 + 52\,(x - 4) + 15\,(x - 4)\,(x - 5) + (x - 4)\,(x - 5)\,(x - 7)$$

$$= x^2\,(x - 1) \qquad \text{\_\_\_(2)}$$

Now substituting x = 2, 8 and 15 in equation (2)

$f(2) = 4\,(2 - 1) = 4$

$f(8) = 64\,(8 - 1) = 448$

$f(15) = 225\,(15 - 1) = 3150$

Q.3. Find the polynomial of the lowest possible degree which assumes the values 3, 12, 15, -21 when x has values 3, 2, 1, -1 respectively.

Ans.: Constructing table according to given data

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

Substituting the values in Newton's divided difference formula :

$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)\,(x - x_1)\ \_\ \_\ \_\ (x - x_{n-1}) + f(x_0, x_1, x_2 \,..x_n)$

$= -21 + \{x - (-1)\}\, 18 + \{x - (-1)\}\,(x - 1)\,(-7) + \{x - (-1)\}\,(x - 1)\,(x - 2)\,(1)$

$= x^3 - 9x^2 + 17x + 6$

□ □ □

Lagrange's Interpolation

LAGRANGE'S INTERPOLATION FORMULA

Let $f(x_0)$, $f(x_1)$,......, $f(x_n)$ be $(n + 1)$ entries of a function $y = f(x)$, where $f(x)$ is assumed to be a polynomial corresponding to the arguments $x_0, x_1, x_2, ..., x_n$.

The polynomial $f(x)$ may be written as

$$f(x) = A_0 (x - x_1)(x - x_2) ...... (x - x_n)$$
$$+ A_1(x - x_0)(x - x_2) ...... (x - x_n)$$
$$+ ...... + A_n (x - x_0)(x - x_1) ...... (x - x_{n-1})$$

where $A_0, A_1, ..., A_n$ are constants to be determined.

Putting $x = x_0, x_1, ..., x_n$ in **eq** we get

$$f(x_0) = A_0 (x_0 - x_1)(x_0 - x_2) .... (x_0 - x_n)$$

$$\therefore \quad A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2) ...... (x_0 - x_n)}$$

$$f(x_1) = A_1 (x_1 - x_0)(x_1 - x_2) ...... (x_1 - x_n)$$

$$\therefore \qquad A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n)}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Similarly, $A_n = \dfrac{f(x_n)}{(x_n - x_0)(x_n - x_1)\ldots\ldots(x_n - x_{n-1})}$

Substituting the values of A_0, A_1, ..., A_n in equation , we get

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n)} f(x_0)$$
$$+ \frac{(x - x_0)(x - x_2)\ldots\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n)} f(x_1)$$
$$+ \ldots\ldots + \frac{(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots\ldots(x_n - x_{n-1})} f(x_n)$$

This is called **Lagrange's Interpolation Formula.** In eqn. (54), dividing both sides by $(x - x_0)(x - x_1) \ldots\ldots (x - x_n)$, Lagrange's formula may also be written as

$$\frac{f(x)}{(x-x_0)(x-x_1)\ldots\ldots(x-x_n)} = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} \cdot \frac{1}{(x-x_0)}$$

$$+ \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} \cdot \frac{1}{(x-x_1)} + \ldots\ldots$$

$$+ \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})} \cdot \frac{1}{(x-x_n)}.$$

Another form of Lagrange's Formula

§ **Prove that** the Lagrange's formula can be put in the form

$$\boxed{P_n(x) = \sum_{r=0}^{n} \frac{\phi(x)\, f(x_r)}{(x-x_r)\,\phi'(x_r)}}$$

where $\phi(x) = \prod_{r=0}^{n} (x - x_r)$

and $\phi'(x_r) = \left[\frac{d}{dx}\{\phi(x)\}\right]_{x = x_r}$

We have the Lagrange's formula,

$$P_n(x) = \sum_{r=0}^{n} \frac{(x-x_0)(x-x_1)\dots(x-x_{r-1})(x-x_{r+1})\dots(x-x_n)}{(x_r-x_0)(x_r-x_1)\dots(x_r-x_{r-1})(x_r-x_{r+1})\dots(x_r-x_n)} f(x_r)$$

$$= \sum_{r=0}^{n} \left\{\frac{\phi(x)}{x-x_r}\right\} \left\{\frac{f(x_r)}{(x_r-x_0)(x_r-x_1)\dots(x_r-x_{r-1})(x_r-x_{r+1})\dots(x_r-x_n)}\right\}$$

Now,

$$\phi(x) = \prod_{r=0}^{n} (x-x_r)$$

$$= (x-x_0)(x-x_1) \dots\dots (x-x_{r-1})(x-x_r)(x-x_{r+1}) \dots\dots (x-x_n)$$

$$\therefore \quad \phi'(x) = (x-x_1)(x-x_2) \dots\dots (x-x_r) \dots\dots (x-x_n)$$

$$+ (x-x_0)(x-x_2) \dots\dots (x-x_r) \dots\dots (x-x_n) + \dots\dots$$

$$+ (x-x_0)(x-x_1) \dots\dots (x-x_{r-1})(x-x_{r+1}) \dots\dots (x-x_n) + \dots\dots$$

$$+ (x-x_0)(x-x_1) \dots\dots (x-x_r) \dots\dots (x-x_{n-1})$$

$$\Rightarrow \quad \phi'(x_r) = [\phi'(x)]_{x=x_r}$$

$$= (x_r-x_0)(x_r-x_1) \dots\dots (x_r-x_{r-1})(x_r-x_{r+1}) \dots\dots (x_r-x_n)$$

Hence from eq

$$P_n(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x-x_r)\phi'(x_r)}$$

Q.1 *Using Lagrange's interpolation formula, find y(10) from the following table:*

x	5	6	9	11
y	12	13	14	16

Ans

Here $x_0 = 5, \quad x_1 = 6, \quad x_2 = 9, \quad x_3 = 11$

$f(x_0) = 12, \quad f(x_1) = 13, \quad f(x_2) = 14, \quad f(x_3) = 16$

Lagrange's formula is

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} (12)$$

$$+ \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} (13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} (14)$$

$$+ \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} (16)$$

$$= -\frac{1}{2}(x-6)(x-9)(x-11) + \frac{13}{15}(x-5)(x-9)(x-11)$$

$$-\frac{7}{12}(x-5)(x-6)(x-11)$$

$$+\frac{4}{15}(x-5)(x-6)(x-9)$$

Putting $x = 10$, we get

$$f(10) = -\frac{1}{2}(10-6)(10-9)(10-11) + \frac{13}{15}(10-5)(10-9)(10-11)$$

$$-\frac{7}{12}(10-5)(10-6)(10-11) + \frac{4}{15}(10-5)(10-6)(10-9)$$

$$= 14.66666667$$

Hence,

$$y(10) = 14.66666667.$$

Q.2 *By means of Lagrange's formula, prove that*

$$y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

Ans

For the arguments $-3, -1, 1, 3$, the Lagrange's formula is

$$y_x = \frac{(x+1)(x-1)(x-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(x+3)(x-1)(x-3)}{(-1+3)(-1-1)(-1-3)} y_{-1}$$

$$+ \frac{(x+3)(x+1)(x-3)}{(1+3)(1+1)(1-3)} y_1$$

$$+ \frac{(x+3)(x+1)(x-1)}{(3+3)(3+1)(3-1)} y_3$$

$$= \frac{(x+1)(x-1)(x-3)}{(-48)} y_{-3} + \frac{(x+3)(x-1)(x-3)}{16} y_{-1}$$

$$+ \frac{(x+3)(x+1)(x-3)}{(-16)} y_1$$

$$+ \frac{(x+3)(x+1)(x-1)}{48} y_3$$

Given $x = 0$, we get

$$y_0 = -\frac{1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

Q.3. Given that

$f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(4) = 16$, $f(7) = 128$

Find the value of $f(5)$ with the help of Lagrange's interpolation formula.

Ans.: According to question

$x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 7$, and

$f(x_0) = 2$, $f(x_1) = 4$, $f(x_2) = 8$, $f(x_3) = 16$, and $f(x_4) = 128$,

Using Lagrange's formula for $x = 5$

$$f(5) = \frac{(5-2)(5-3)(5-4)(5-7)}{(1-2)(1-3)(1-4)(1-7)} \times 2 + \frac{(5-1)(5-3)(5-4)(5-7)}{(2-1)(2-3)(2-4)(2-7)} \times 4$$

$$+ \frac{(5-1)(5-2)(5-4)(5-7)}{(3-1)(3-2)(3-4)(3-7)} \times 8 + \frac{(5-1)(5-2)(5-3)(5-7)}{(4-1)(4-2)(4-3)(4-7)} \times 16$$

$$+ \frac{(5-1)(5-2)(5-3)(5-4)}{(7-1)(7-2)(7-3)(7-4)} \times 128$$

$$= \frac{-2}{3} + \frac{32}{5} - 24 + \frac{128}{3} + \frac{128}{15} = \frac{494}{15}$$

$= 32.93333$

Hence $f(5) = 32.9333$

Q.4. Find the form of function given by the following table :

x	**3**	**2**	**1**	**-1**
$\boldsymbol{f}$**(x)**	**3**	**12**	**15**	**-21**

Ans.: According to question

$x_0 = 3$, $x_1 = 2$, $x_2 = 1$ and $x_3 = -1$

$f(x_0) = 2$, $f(x_1) = 12$, $f(x_2) = 15$ and $f(x_3) = -21$

Now substituting above values in Lagrange's formula :

$$f(x) = \frac{(x-2)(x-1)(x+1)}{(3-2)(3-1)(3+1)} \times 3 + \frac{(x-3)(x-1)(x+1)}{(2-3)(2-1)(2+1)} \times 12$$

$$+ \frac{(x-3)(x-2)(x+1)}{(1-3)(1-2)(1+1)} \times 15 + \frac{(x-3)(x-2)(x-1)}{(-1-3)(-1-2)(-1-1)} \times -21$$

$$= \frac{3}{8}(x^3 - 2x^2 - x + 2) - 4(x^3 - 3x^2 - x + 3) + \frac{15}{4}(x^3 - 4x^2 + x + 6) + \frac{7}{8}(x^3 - 6x^2 + 11x - 6)$$

$$f(x) = x^3 - 9x^2 + 17x + 6$$

Q.5. By means of Lagrange's formula prove that :

$$\mathbf{y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]}$$

Ans.: Here we are given y_{-3}, y_{-1} y_1 and y_3 and we have to evaluate y_0.

Using Lagrange's formula

$$y_0 = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)} y_{-1}$$

$$+ \frac{(0+3)(0+1)(0-3)}{(1+3)(1+1)(1-3)} y_1 + \frac{(0+3)(0+1)(0-3)}{(3+3)(3+1)(3-1)} y_3$$

$$= \frac{-1}{16}\ y_{-3} + \frac{9}{16}\, y_{-1} + \frac{9}{16}\, y_1 - \frac{1}{16}\, y_3$$

$$= \frac{1}{2}\ \left(y_1 + y_{-1}\right) - \frac{1}{16}\ \left(y_3 - y_1\right) - \left(y_{-1} - y_{-3}\right)$$

$$= \frac{1}{2}\ \left(y_1 + y_{-1}\right) - \frac{1}{8}\left[\frac{1}{2}\left(y_3 - y_1\right) - \frac{1}{2}\left(y_{-1} - y_{-3}\right)\right]$$

□

Unit 2

Central Difference

Central differences. The central difference operator d is defined by the relations

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2},\ \ldots\ldots,\ y_n - y_{n-1} = \delta y_{n-\frac{1}{2}}.$$

Similarly, high order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \quad \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2$$

and so on.

These differences are shown as follows:

Central difference table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

Gauss Forward Interpolation Formula

By Newton Interpolation Formula

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{1 \times 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{1 \times 2 \times 3}\Delta^3 y_0 + \ldots, \tag{1}$$

From central difference table we have

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \ldots$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

$$\vdots$$

Substituting the values in (1) we get

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{u(u-1)(u-2)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) + \frac{u(u-1)(u-2)(u-3)}{4!}\left(\Delta^4 y_{-1} + \Delta^5 y_{-1}\right) + \ldots$$

The above formula may be written as

$$y_4 = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots \tag{2}$$

This is called Gauss Forward Interpolation Formula.

Gauss Backward Interpolation Formula

We have

$$\Delta y_0 = \Delta y_0 + \Delta^2 y_{-1}$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\vdots$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^2 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

By Newton Forward Formula

$$y = f(x) = y_0 + \frac{u}{1!}\left(\Delta y_{-1} + \Delta^2 y_{-1}\right) + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \ldots$$

$$\Rightarrow\ y_4 = y_0 \frac{u}{1!}\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} +$$

$$\frac{(u+2)\,(u+1)\,u\,(u-1)}{4!}\Delta^4 y_{-2} + \ldots$$

This is called Gauss Backward Interpolation Formula.

Stirling's Formula

Gauss Forward Interpolation Formula

$$y_u = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} +$$

$$\frac{(u+1)u(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \ldots$$

Gauss Backward Interpolation Formula

$$y_u = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{(u-1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} +$$

$$\frac{(u+2)u(u+1)(u-1)}{4!}\Delta^4 y_{-2} + \ldots$$

Taking the mean of these formula

$$y_u = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2}\Delta^2 y_{-1} +$$

$$\frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-4)}{5!}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \ldots$$

Q.1 Apply the central difference formula to obtain f(32)

$f(25) = 0.2707$ $\quad f(35) = 0.3386$

$f(30) = 0.3027$ $\quad f(40) = 0.3794$.

Ans

U= (32- 30)/5 = 0.4

The forward difference table is:

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
−1	25	.2707			
			.032		
0	30	.3027		.0039	
			.0359		.0010
1	35	.3386		.0049	
			.0408		
2	40	.3794			

Applying Gauss' forward difference formula, we have

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-1)$$

$$\therefore \quad f(.4) = .3027 + (.4)(.0359) + \frac{(.4)(.4-1)}{2!}(.0039) + \frac{(1.4)(.4)(.4-1)}{3!}(.0010)$$

$$= 0.316536.$$

Q.2

Find the value of cos 51° 42′ by Gauss's backward formula.

Given that

x:	50°	51°	52°	53°	54°
$\cos x$:	0.6428	0.6293	0.6157	0.6018	0.5878.

Ans

Taking the origin at 52° and $h = 1$

$$\therefore \quad u = (x - a) = 51°\ 42' - 52° = -\ 18' = -\ 0.3°$$

Gauss's backward formula is

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\Delta^3 f(-2)$$

$$+ \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 f(-2) \quad (39)$$

The difference table is as below:

u	x	$10^4\ f(x)$	$10^4\ \Delta\ f(x)$	$10^4\ \Delta^2\ f(x)$	$10^4\ \Delta^3\ f(x)$	$10^4\ \Delta^4\ f(x)$
– 2	50°	6428				
			– 135			
– 1	51°	6293		– 1		
			– 136		– 2	
0	52°	6157		– 3		4
			– 139		2	
1	53°	6018		– 1		
			– 140			
2	54°	5878				

From (39),

$$10^4 f(-\ .3) = 6157 + (-\ .3)(-\ 136) + \frac{(.7)(-\ .3)}{2!}(-\ 3) + \frac{(.7)(-\ .3)(-\ 1.3)}{3!}(-\ 2)$$

$$+ \frac{(1.7)(.7)(-\ .3)(-\ 1.3)}{4!}(4)$$

$$= 6198.10135$$

$$\therefore \quad f(-\ .3) = .619810135$$

Hence $\cos 51°42' = 0.619810135.$

Q.3

Use Stirling's formula to find y_{28}, given

$y_{20} = 49225,\quad y_{25} = 48316,\quad y_{30} = 47236,$

$y_{35} = 45926,\quad y_{40} = 44306.$

Ans

Let the origin be at 30 and $h = 5$

$$a + hu = 28$$

$$\Rightarrow \quad 30 + 5u = 28 \quad \Rightarrow \quad u = -.4$$

The difference table is as follows:

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
– 2	20	49225				
			– 909			
– 1	25	48316		– 171		
			– 1080		– 59	
0	30	47236		– 230		– 21
			– 1310		– 80	
1	35	45926		– 310		
			– 1620			
2	40	44306				

By Stirling's formula,

$$f(-.4) = 47236 + (-.4)\left(\frac{-1080-1310}{2}\right) + \frac{(-.4)^2}{2!}(-230)$$

$$+ \frac{(.6)(-.4)(-1.4)}{3!}\left(\frac{-59-80}{2}\right) + \frac{(-.4)^2\{(-.4)^2-1\}}{4!}(-21)$$

$$= 47691.8256$$

Hence $\quad y_{28} = 47691.8256.$

Q.

Given $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$ and $y_{32} = 40$ find y_{25} by Bessel's interpolation formula.

Ans

Take origin at 24.

Here, $\quad a = 24, \quad h = 4, \quad a + hu = 25$

$$\therefore \quad 24 + 4u = 25 \quad \Rightarrow \quad u = .25$$

The difference table is:

u	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
−1	20	24			
			8		
0	24	32		−5	
			3		7
1	28	35		2	
			5		
2	32	40			

Using Bessel's formula,

$$f(u) = \left\{\frac{f(0)+f(1)}{2}\right\} + \left(u - \frac{1}{2}\right)\Delta f(0)$$

$$+ \frac{u(u-1)}{2}\left\{\frac{\Delta^2 f(-1) + \Delta^2 f(0)}{2}\right\}$$

$$+ \frac{(u-1)\left(u-\frac{1}{2}\right)u}{3!}\Delta^3 f(-1)$$

$$\Rightarrow \quad f(.25) = \left(\frac{32+35}{2}\right) + (.25-.5)(3) + \frac{(.25)(.25-1)}{2}\left\{\frac{-5+2}{2}\right\}$$

$$+ \frac{(.25-1)(.25-.5)(.25)}{3!}(7)$$

$$= 32.9453125$$

Hence $y_{25} = 32.9453125$.

Q.2 *If third differences are constant, prove that*

$$y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x)$$

Ans

. Putting $u = \frac{1}{2}$ in Bessel's formula, we get

$$y_{1/2} = \frac{1}{2}(y_0 + y_1) - \frac{1}{16}(\Delta^2 y_0 + \Delta^2 y_{-1})$$

Shifting the origin to x,

$$y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_x + \Delta^2 y_{x-1}).$$

Q.3

Given $y_0, y_1, y_2, y_3, y_4, y_5$ *(fifth differences constant), prove that*

$$y_{2\frac{1}{2}} = \frac{1}{2}c + \frac{25(c-b) + 3(a-c)}{256}$$

where $a = y_0 + y_5, b = y_1 + y_4, c = y_2 + y_3.$

Ans

. Put $u = \frac{1}{2}$ in Bessel's formula, we get

$$y_{1/2} = \frac{1}{2}(y_0 + y_1) - \frac{1}{16}(\Delta^2 y_0 + \Delta^2 y_{-1}) + \frac{3}{256}(\Delta^4 y_{-1} + \Delta^4 y_{-2})$$

Shifting the origin to 2, we have

$$y_{2\frac{1}{2}} = \frac{1}{2}(y_2 + y_3) - \frac{1}{16}(\Delta^2 y_2 + \Delta^2 y_1) + \frac{3}{256}(\Delta^4 y_1 + \Delta^4 y_0)$$

$$= \frac{c}{2} - \frac{1}{16}(y_3 - 2y_2 + y_1 + y_4 - 2y_3 + y_2)$$

$$+ \frac{3}{256}(y_5 - 3y_4 + 2y_3 + 2y_2 - 3y_1 + y_0)$$

$$y_{2\frac{1}{2}} = \frac{c}{2} - \frac{1}{16}(y_4 - y_3 - y_2 + y_1) + \frac{3}{256}(a - 3b + 2c)$$

$$= \frac{c}{2} - \frac{1}{16}(b - c) + \frac{3}{256}(a - 3b + 2c)$$

$$y_{2\frac{1}{2}} = \frac{c}{2} + \frac{1}{256}[25(c - b) + 3(a - c)].$$

Differention

(1) **Newton's forward difference interpolation formula is**

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2\,!}\,\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3\,!}\,\Delta^3 y_0 + \ldots \qquad (1)$$

where $$u = \frac{x-a}{h} \qquad (2)$$

Differentiating eqn. (1) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\,\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\,\Delta^3 y_0 + \ldots \qquad (3)$$

Differentiating eqn. (2) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \qquad (4)$$

We know that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \ldots\right] \qquad (5)$$

Expression (5) provides the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Formula (5) becomes simple for tabulated values of x, in particular when $x = a$ and $u = 0$

Putting $u = 0$ in (5), we get

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \ldots\right] \quad (6)$$

Differentiating eqn. (5) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right]\frac{1}{h}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\,\Delta^3 y_0 + \left(\frac{6u^2 - 18u + 11}{12}\right)\Delta^4 y_0 + \ldots\right] \quad (7)$$

Putting $u = 0$ in (7), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right) \quad (8)$$

Similarly, we get

$$\left(\frac{d^3y}{dx^3}\right)_{x=a} = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right) \quad (9)$$

and so on.

(2) **Newton's backward difference interpolation formula is**

$$y = y_n + u\,\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots \qquad (10)$$

where $u = \dfrac{x - x_n}{h}$ (11)

Differentiating (10) with respect to, u, we get

$$\frac{dy}{du} = \nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2+6u+2}{6}\right)\nabla^3 y_n + \ldots \qquad (12)$$

Differentiating (11) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \qquad (13)$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\nabla y_n + \left(\frac{2u+1}{2}\right)\nabla^2 y_n + \left(\frac{3u^2+6u+2}{6}\right)\nabla^3 y_n + \ldots\right] \qquad (14)$$

Expression (14) provides us the value of $\dfrac{dy}{dx}$ at any x which is not tabulated.

At $x = x_n$, we have $u = 0$

∴ Putting $u = 0$ in (14), we get

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right) \qquad (15)$$

Differentiating (14) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\,\nabla^3 y_n + \left(\frac{6u^2 + 18u + 11}{12}\right)\nabla^4 y_n + \ldots\right] \quad (16)$$

Putting $u = 0$ in (16), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right) \quad (17)$$

Similarly, we get

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right) \quad (18)$$

and so on.

(3) **Stirling's central difference interpolation formula is**

$$y = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots \tag{19}$$

where $$u = \frac{x-a}{h} \tag{20}$$

Differentiating eqn. (19) with respect to u, we get

$$\frac{dy}{du} = \frac{\Delta y_0 + \Delta y_{-1}}{2} + u\,\Delta^2 y_{-1} + \left(\frac{3u^2-1}{6}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+ \left(\frac{4u^3-2u}{4!}\right)\Delta^4 y_{-2} + \left(\frac{5u^4-15u^2+4}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots \tag{21}$$

Differentiating (20) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h} \tag{22}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\,\Delta^2 y_{-1} + \left(\frac{3u^2-1}{6}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$$

$$\left. + \left(\frac{4u^3-2u}{4!}\right)\Delta^4 y_{-2} + \left(\frac{5u^4-15u^2+4}{5!}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right] \tag{23}$$

Expression (23) provides the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Given $x = a$, we have $u = 0$

∴ Given $u = 0$ in (23), we get

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) - \ldots\right] \quad (24)$$

Differentiating (23) with respect to x, we get

$$\frac{d^2 y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{6u^2 - 1}{12}\right)\Delta^4 y_{-2}\right.$$

$$\left. + \left(\frac{2u^3 - 3u}{12}\right)\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots\right] \quad (25)$$

Given $u = 0$ in (25), we get

$$\left(\frac{d^2 y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\right) \quad (26)$$

and so on.

(4) **Bessel's central difference interpolation formula is**

$$y = \left(\frac{y_0 + y_1}{2}\right) + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{u(u-1)\left(u - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{(u+1)\,u(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)$$

$$+ \frac{(u+1)\,u(u-1)(u-2)\left(u - \frac{1}{2}\right)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{(u+2)(u+1)\,u(u-1)(u-2)(u-3)}{6!}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots \quad (27)$$

where $u = \dfrac{x - a}{h}$ (28)

Differentiating eqn. (27) with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \left(\frac{2u-1}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{3u^2 - 3u + \frac{1}{2}}{3!}\right) \Delta^3 y_{-1}$$

$$+ \left(\frac{4u^3 - 6u^2 - 2u + 2}{4!}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \left(\frac{5u^4 - 10u^3 + 5u - 1}{5!}\right) \Delta^5 y_{-2}$$

$$+ \left(\frac{6u^5 - 15u^4 - 20u^3 + 45u^2 + 8u - 12}{6!}\right)\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots \qquad (29)$$

Differentiating (28) with respect to x, we get

$$\frac{du}{dx} = \frac{1}{h}$$

Now, $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2!}\right)\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{3u^2 - 3u + \frac{1}{2}}{3!}\right) \Delta^3 y_{-1}\right.$$

$$+ \left(\frac{4u^3 - 6u^2 - 2u + 2}{4!}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \left(\frac{5u^4 - 10u^3 + 5u - 1}{5!}\right) \Delta^5 y_{-2}$$

$$\left.+ \left(\frac{6u^5 - 15u^4 - 20u^3 + 45u^2 + 8u - 12}{6!}\right)\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right] \qquad (30)$$

Expression (30) provides us the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Given $x = a$, we have $u = 0$

∴ Given $u = 0$ in (30), we get

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\right.$$

$$\left. - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{60}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right] \qquad (31)$$

Differentiating (30) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{2u-1}{2}\right)\Delta^3 y_{-1} + \left(\frac{6u^2-6u-1}{12}\right)\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\right.$$

$$+ \left(\frac{4u^3-6u^2+1}{24}\right)\Delta^5 y_{-2}$$

$$\left. + \left(\frac{15u^4 - 30u^3 - 30u^2 + 45u + 4}{360}\right)\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right] \quad (32)$$

Given $u = 0$ in (32), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left[\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) - \frac{1}{2}\Delta^3 y_{-1} - \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right)\right.$$

$$\left. + \frac{1}{24}\Delta^5 y_{-2} + \frac{1}{90}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right] \quad (33)$$

and so on.

(5) **For unequally spaced values of the argument**

(*i*) **Newton's divided difference formula is**

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)$$

$$(x - x_2)\,\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0) + \ldots \quad (34)$$

$f'(x)$ is given by

$$f'(x) = \Delta f(x_0) + \{2x - (x_0 + x_1)\}\,\Delta^2 f(x_0) + \{3x^2 - 2x(x_0 + x_1 + x_2)$$

$$+ (x_0x_1 + x_1x_2 + x_2x_0)\}\,\Delta^3 f(x_0) + \ldots \quad (35)$$

(*ii*) **Lagrange's interpolation formula is**

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} f(x_1) + \ldots \quad (36)$$

$f'(x)$ can be obtained by differentiating $f(x)$ in eqn. (36).

1. *Formula (8) can be extended as*

$$\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 - \frac{7}{10}\Delta^7 + \frac{363}{560}\Delta^8 + \ldots\right) y_0$$

2. *Formula (17) can be extended as*

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \frac{137}{180}\nabla^6 + \frac{7}{10}\nabla^7 + \frac{363}{560}\nabla^8 + \ldots\right) y_n.$$

Q.1

From the following table of values of x and y, obtain $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *for x = 1.2, 2.2 and 1.6*

x:	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250.

Ans

The forward difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

(*i*) Here $a = 1.2$

$\therefore \quad y_0 = 3.3201; \quad h = 0.2$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.008) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{(0.2)^2}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right]$$

$$= 3.318$$

(*ii*) Here $a = 2.2$,

$\therefore \quad y_n = 9.02 \quad \text{and} \quad h = 0.2$

$$\left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014)\right]$$

$$= 9.0228$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014)\right]$$

$$= 8.992.$$

(iii) Here $a = 1.6$

$\therefore$ $y_0 = 4.9530,\ y_{-1} = 4.0552$

$y_{-2} = 3.3201,\ y_{-3} = 2.7183$ and $h = 0.2$

By using Stirling's formula for derivatives, we get

$$\left[\frac{dy}{dx}\right]_{x=1.6} = \frac{1}{0.2}\left[\left(\frac{1.0966 + 0.8978}{2}\right) - \frac{1}{6}\left(\frac{0.0441 + 0.0361}{2}\right) + \frac{1}{30}\left(\frac{0.0014 + 0.0013}{2}\right)\right]$$

$$= 4.9530$$

and
$$\left[\frac{d^2y}{dx^2}\right]_{x=1.6} = \frac{1}{0.04}\left[0.1988 - \frac{1}{12}(.0080) + \frac{1}{90}(.0001)\right]$$

$$= 4.9525.$$

Q.2

Using Bessel's formula, find f ′(7.5) from the following table:

x:	*7.47*	*7.48*	*7.49*	*7.5*	*7.51*	*7.52*	*7.53*
f(x):	*0.193*	*0.195*	*0.198*	*0.201*	*0.203*	*0.206*	*0.208.*

Ans

The difference table is:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
7.47	0.193						
		0.002					
7.48	0.195		0.001				
		0.003		– 0.001			
7.49	0.198		0.000		0.000		
		0.003		– 0.001		0.003	
7.50	0.201		– 0.001		0.003		– 0.01
		0.002		0.002		– 0.007	
7.51	0.203		0.001		– 0.004		
		0.003		– 0.002			
7.52	0.206		– 0.001				
		0.002					
7.53	0.208						

Let $a = 7.5$, $h = 0.01$

$$f'(7.5) = \left(\frac{dy}{dx}\right)_{x=7.5} = \frac{1}{0.01}\left[\Delta y_0 - \frac{1}{2}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{1}{12}\Delta^3 y_{-1}\right.$$

$$\left. + \frac{1}{12}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{60}\left(\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right) + \ldots\right]$$

$$= \frac{1}{0.01}\left[(.002) - \frac{1}{2}\left\{\frac{-.001 + .001}{2}\right\} + \frac{1}{12}(0.002) + \frac{1}{12}\right.$$

$$\left.\left\{\frac{.003 + (-.004)}{2}\right\} - \frac{1}{120}(-0.007) - \frac{1}{60}\left(\frac{-.01}{2}\right)\right]$$

$$= 0.226667.$$

Q.

Assuming Bessel's interpolation formula, prove that

$$\frac{d}{dx}(y_x) = \Delta y_{x-1/2} - \frac{1}{24}\Delta^3 y_{x-3/2} + \ldots$$

Ans

Bessel's formula is

$$y_x = \left(\frac{y_0 + y_1}{2}\right) + \left(x - \frac{1}{2}\right)\Delta y_0 + \frac{x(x-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{x(x-1)\left(x - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \ldots$$

Replacing x by $x + \frac{1}{2}$, we get

$$y_{x+1/2} = \left(\frac{y_0 + y_1}{2}\right) + x\,\Delta y_0 + \frac{\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)x}{3!}\Delta^3 y_{-1} + \ldots$$

Differentiating (47) with respect to x, we get

$$\frac{d}{dx}(y_{x+1/2}) = \Delta y_0 + \frac{2x}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \left(\frac{3x^2 - \frac{1}{4}}{3!}\right)\Delta^3 y_{-1} + \ldots$$

Given $x = 0$, we get

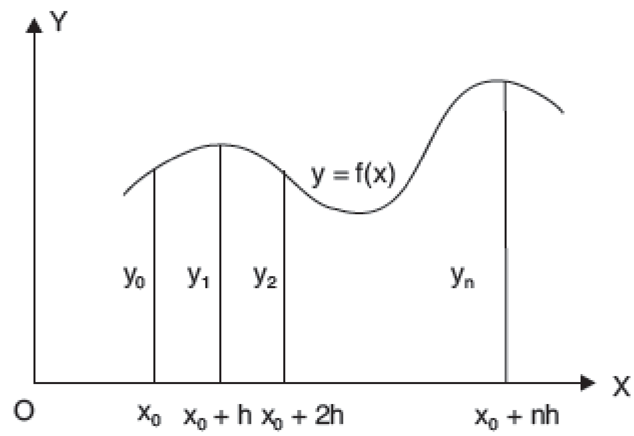
$$\frac{d}{dx}(y_{x+1/2}) = \Delta y_0 - \frac{1}{24}\,\Delta^3 y_{-1} + \ldots$$

Shifting the origin from $x = 0$ to $x - \frac{1}{2}$, we get

$$\frac{d}{dx}(y_x) = \Delta y_{x-1/2} - \frac{1}{24}\Delta^3 y_{x-3/2} + \ldots$$

NUMERICAL INTEGRATION

Given a set of tabulated values of the integrand $f(x)$, determining the value of $\int_{x_0}^{x_n} f(x)\,dx$ is called numerical integration. The given interval of integration is subdivided into a large number of subintervals of equal width h and the function tabulated at the points of subdivision is replaced by any one of the interpolating polynomials like Newton-Gregory's, Stirling's, Bessel's over each of the subintervals and the integral is evaluated. There are several formulae for numerical integration which we shall derive in the sequel.



NEWTON-COTE'S QUADRATURE FORMULA

Let $I = \int_a^b y\,dx$, where y takes the values $y_0, y_1, y_2, \ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$.

Let the interval of integration (a, b) be divided into n equal sub-intervals, each of width $h = \frac{b-a}{n}$ so that

$$x_0 = a,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ \ldots\ldots,\ x_n = x_0 + nh = b.$$

$$\therefore \qquad I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

Since any x is given by $x = x_0 + rh$ and $dx = hdr$

$$\therefore \qquad I = h\int_0^n f(x_0 + rh)\,dr$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right] dr$$

[by Newton's forward interpolation formula]

$$= h\left[ry_0 + \frac{r^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{r^3}{3} - \frac{r^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{r^4}{4} - r^3 + r^2\right)\Delta^3 y_0 + \ldots\ldots \right]_0^n$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots \right] \qquad (49)$$

This is a **general quadrature formula** and is known as **Newton-Cote's quadrature formula.** A number of important deductions *viz.* Trapezoidal rule, Simpson's one-third and three-eighth rules, Weddle's rule can be immediately deduced by putting $n = 1, 2, 3,$ and 6, respectively, in formula (49).

TRAPEZOIDAL RULE (n = 1)

Putting $n = 1$ in formula (49) and taking the curve through (x_0, y_0) and (x_1, y_1) as a polynomial of degree one so that differences of an order higher than one vanish, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0 \right) = \frac{h}{2}\,[2y_0 + (y_1 - y_0)] = \frac{h}{2}(y_0 + y_1)$$

Similarly, for the next sub-interval $(x_0 + h, x_0 + 2h)$, we get

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2), \ldots\ldots, \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

which is known as **Trapezoidal rule.** By increasing the number of subintervals, thereby making h very small, we can improve the accuracy of the value of the given integral.

SIMPSON'S ONE-THIRD RULE (n = 2)

Putting $n = 2$ in formula (49) and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{2h}{6}[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)]$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4), \ldots\ldots,$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})]$$

which is known as **Simpson's one-third rule.**

While using this formula, the given interval of integration must be divided into an even number of sub-intervals, since we find the area over two sub-intervals at a time.

SIMPSON'S THREE-EIGHTH RULE (n = 3)

Putting $n = 3$ in formula (49) and taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3) as a polynomial of degree three so that differences of order higher than three vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\,[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}\,[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, $\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_3 + 3y_4 + 3y_5 + y_6]$, ...

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\,[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\,[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})]$$

which is known as **Simpson's three-eighth rule.**

While using this formula, the given interval of integration must be divided into sub-intervals whose number n is a multiple of 3.

Q.1. Compute the value of following integral by Trapezoidal rule.

$$\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,dx$$

Ans.: Dividing the range of integration in equal intervals in the interval [0.2, 1.4]

$$\frac{1.4-0.2}{6}=\frac{1.2}{6}=0.2=h$$

x	sin x	$\log_e x$	e^x	y = sin x – log + e^x
0.2	0.19867	-1.6095	1.2214	y_0 = 3.0296
0.4	0.3894	-0.9163	1.4918	y_1 = 2.7975
0.6	0.5646	-0.5108	1.8221	y_2 = 2.8975
0.8	0.7174	-0.2232	2.2255	y_3 = 3.1661
1.0	0.8415	0.0000	2.7183	y_4 = 3.5598
1.2	0.9320	0.1823	3.3201	y_5 = 4.0698
1.4	0.9855	0.3365	4.0552	y_6 = 4.7042

Using following trapezoidal rule

$$I = \int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx$$

$$= \frac{h}{2}[(y_0 + y_6) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{0.2}{2}[7.7338 + 2\,(16.4907)]$$

$$= 4.07152$$

Q.2. Calculate the value of the integral $\int_{4}^{5.2} \log_e x\, dx$ by Simpson's '$\frac{1}{3}$' rule.

Ans.: First of all dividing the interval [4 5.2] in equal parts.

$$\frac{5.2-4}{6}=\frac{1.2}{6}=0.2=h$$

x_i	$y_i = \log_e x = \log_{10} x \times 2.30258$
4.0	y_0 = 1.3862944
4.2	y_1 = 1.4350845
4.4	y_2 = 1.4816045
4.6	y_3 = 1.5260563
4.8	y_4 = 1.5686159
5.0	y_5 = 1.6049379

5.2	$y_6 = 1.6486586$

Using following Simpson's '$\frac{1}{3}$' rule :

$$I = \frac{h}{3}[(y_0 + y_6) + 4\,(y_1 + y_3 + y_{5)}\ +2\,(y_2 + y_4)]$$

$$= \frac{0.2}{3}[3.034953 + 18.232315 + 6.1004408]$$

$$= \frac{0.2}{3}[27.417709] = 1.8278472$$

Q.3. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's '$\frac{3}{8}$' rule :

Ans.: Dividing the interval [0, 1] into six equal intervals.

$$h = \frac{1-0}{6} = \frac{1}{6}$$

x	**y =** $h = \frac{1}{(1+\mathbf{x}^2)}$
$x_0 = 0$	$y_0 = 1.000$
$x_0 + h = 1/6$	$y_1 = (36/37) = 0.97297$
$x_0 + 2h = 2/6$	$y_2 = (36/40) = 0.90000$
$x_0 + 3h = 3/6$	$y_3 = (36/45) = 0.80000$
$x_0 + 4h = 4/6$	$y_4 = (36/52) = 0.69231$
$x_0 + 5h = 5/6$	$y_1 = (36/61) = 0.59016$
$x_0 + 6h = 1$	$y_6 = (1/2) = 0.50000$

Using following Simpson's '3/8' rule.

$$\int_{x_0}^{x_0+nh} ydx = \frac{3h}{8}\ (y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots) + 2(y_3 + y_6 + \ldots\ldots.)$$

$$\int_0^1 ydx = \frac{1}{16}\quad (1 + 0.5) + 3\,(0.97297 + 0.9 + 0.69231 + 0.59016) + 2\,(0.8)$$

$$= \frac{1}{16} [1.5 + 9.46632 + 1.6] \quad = 0.785395$$

□ □ □

Gauss two point

Unit 3

Bisection Method

Q.1. Find real root of the equation $x^3 - 5x + 3$ upto three decimal digits.

Ans.: Here $f(x) = x^3 – 5x + 3$

$f(0) = 0 – 0 + 3 = 3\ f(x_0)$ (say)

$f(1) = 1 – 5 + 3 = – 1 = f(x_1)$ (say)

Since $f(x_0), f(x_1) < 0$ so the root of the given equation lies between 0 and 1

So, $x_2 = \frac{x_0 + x_1}{2} = \frac{0+1}{2} = 0.5$

Now, $f(x_2) = f(0.5)$

$= (0.5)^3 – 5 (0.5) + 3$

$= 0.125 – 2.5 + 3$

$= 0.625$ (which is positive)

$\because$ $f(x_1).f(x_2) < 0$

So, $x_3 = \frac{x_1 + x_2}{2} = \frac{1 + 0.5}{2} = 0.75$

Now, $f(x_3) = f(0.75)$

$= (0.75)^3 – 5 (0.75) + 3$

$= 0.4218 – 3.75 + 3$

$= – 0.328$ (which is negative)

$\because$ $f(x_2).f(x_3) < 0$

So, $x_4 = \frac{x_2 + x_3}{2} = \frac{0.5+0.75}{2} = 0.625$

Now, $f(x_4) = f(0.625)$

$= (0.625)^3 – 5 (0.625) + 3$

$= 0.244 – 3.125 + 3$

$= 0.119$ (which is positive)

$\because$ $f(x_3).f(x_4) < 0$

So, $x_5 = \frac{x_3 + x_4}{2} = \frac{0.75 + 0.625}{2} = 0.687$

Now, $f(x_5) = f(0.687)$

$= (0.687)^3 - 5\,(0.687) + 3$

$= -\,0.1108$ (which is negative)

$\because \quad f(x_4).f(x_5) < 0$

So, $x_6 = \frac{x_4 + x_5}{2} = \frac{0.625 + 0.687}{2} = 0.656$

Now, $f(x_6) = f(0.656)$

$= (0.656)^3 - 5\,(0.656) + 3$

$= 0.0023$ (which is positive)

$\because \quad f(x_5).f(x_6) < 0$

So, $x_7 = \frac{x_5 + x_6}{2} = \frac{0.687 + 0.656}{2} = 0.671$

Now, $f(x_7) = f(0.671)$

$= (0.671)^3 - 5\,(0.671) + 3$

$= -\,0.0528$ (which is negative)

$\because \quad f(x_6).f(x_7) < 0$

So, $x_8 = \frac{x_6 + x_7}{2} = \frac{0.656 + 0.671}{2} = 0.663$

Now, $f(x_8) = f(0.663)$

$= (0.663)^3 - 5\,(0.663) + 3$

$= 0.2920 - 3.315 + 3$

$= -\,0.023$ (which is negative)

$\because \quad f(x_6).f(x_8) < 0$

So, $x_9 = \frac{x_6 + x_8}{2} = \frac{0.656 + 0.663}{2} = 0.659$

Now, $f(x_9) = f(0.659)$

$= (0.659)^3 - 5\,(0.659) + 3$

$= -\,0.0089$ (which is negative)

$\because \quad f(x_6).f(x_9) < 0$

So, $x_{10} = \frac{x_6 + x_9}{2} = \frac{0.656 + 0.659}{2} = 0.657$

Now, $f(x_{10}) = f(0.657)$

$= (0.657)^3 - 5\,(0.657) + 3$

$= -\,0.00140$ (which is negative)

$\because \quad f(x_6).f(x_{10}) < 0$

So, $x_{11} = \frac{x_6 + x_{10}}{2} = \frac{0.656 + 0.657}{2} = 0.656$

Now, $f(x_{11}) = f(0.656)$

$= (0.656)^3 - 5\,(0.656) + 3$

$= 0.2823 - 3.28 + 3$

$= 0.00230$ (which is positive)

$\because \quad f(x_{11}).f(x_{10}) < 0$

So, $x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{0.657 + 0.656}{2} = 0.656$

Since x_{11} and x_{12} both same value. Therefore if we continue this process we will get same value of x so the value of x is 0.565 which is required result.

Q.2. Find real root of the equation $\cos x - xe^x = 0$ correct upto four decimal places.

Ans.: Since, $f(x) = \cos x - xe^x$

So, $f(0) = \cos 0 - 0e^0 = 1$ (which is positive)

And $f(1) = \cos 1 - 1e^1 = -2.1779$ (which is negative)

$\because \quad f(0).f(1) < 0$

Hence the root of are given equation lies between 0 and 1.

let $f(0) = f(x_0)$ and $f(1) = f(x_1)$

So, $x_2 = \frac{x_0 + x_1}{2} = \frac{0 + 1}{2} = 0.5$

Now, $f(x_2) = f(0.5)$

$f(0.5) = \cos(0.5) - (0.5)e^{0.5}$

$= 0.05322$ (which is positive)

$\because \quad f(x_1).f(x_2) < 0$

So, $x_3 = \frac{x_1 + x_2}{2} = \frac{1 + 0.5}{2} = \frac{1.5}{2} = 0.75$

Now, $f(x_3) = f(0.75)$

$= \cos(0.75) - (0.75)e^{0.75}$

$= -0.856$ (which is negative)

$\because \quad f(x_2).f(x_3) < 0$

So, $x_4 = \frac{x_2 + x_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$

$f(x_4) = f(0.625)$

$= \cos(0.625) - (0.625)e^{(0.625)}$

$= -0.356$ (which is negative)

$\because \quad f(x_2).f(x_4) < 0$

So, $x_5 = \frac{x_2 + x_4}{2} = \frac{0.5 + 0.625}{2} = 0.5625$

Now, $f(x_3) = f(0.5625)$

$= \cos(0.5625) - 0.5625e^{0.5625}$

$= -0.14129$ (which is negative)

$\because \quad f(x_2).f(x_5) < 0$

So, $x_6 = \frac{x_2 + x_5}{2} = \frac{0.5 + 0.5625}{2} = 0.5312$

Now, $f(x_6) = f(0.5312)$

$= \cos(0.5312) - (0.5312)e^{0.5312}$

$= -0.0415$ (which is negative)

$\because \quad f(x_2).f(x_6) < 0$

So, $x_7 = \frac{x_2 + x_6}{2} = \frac{0.5 + 0.5312}{2} = 0.5156$

Now, $f(x_7) = f(0.5156)$

$= \cos(0.5156) - (0.5156)e^{0.5156}$

$= 0.006551$ (which is positive)

$\because \quad f(x_6).f(x_7) < 0$

So, $x_8 = \frac{x_6 + x_7}{2} = \frac{0.513 + 0.515}{2} = 0.523$

Now, $f(x_8) = f(0.523)$

$= \cos(0.523) - (0.523)e^{0.523}$

$= -0.01724$ (which is negative)

$\because \quad f(x_7).f(x_8) < 0$

So, $(x_9) = \frac{x_7 + x_8}{2} = \frac{0.515 + 0.523}{2} = 0.519$

Now, $f(x_9) = f(0.519)$

$= \cos(0.519) - (0.519)e^{0.519}$

$= -0.00531$ (which is negative)

$\because \quad f(x_7).f(x_9) < 0$

So, $(x_{10}) = \frac{x_7 + x_9}{2} = \frac{0.515 + 0.519}{2} = 0.5175$

Now, $f(x_{10}) = f(0.5175)$

$= \cos(0.5175) - (0.5175)e^{0.5175}$

$= 0.0006307$ (which is positive)

$\because \quad f(x_9).f(x_{10}) < 0$

So, $x_{11} = \frac{x_9 + x_{10}}{2} = \frac{0.5195 + 0.5175}{2} = 0.5185$

Now, $f(x_{11}) = f(0.5185)$

$= \cos(0.5185) - (0.5185)e^{0.5185}$

$= -0.002260$ (which is negative)

$\because \quad f(x_{10}).f(x_{11}) < 0$

So, $x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{0.5175 + 0.5185}{2} = 0.5180$

Hence the root of the given equation upto 3 decimal places is x = 0.518

Thus the root of the given equation is x = 0.518

□

Regula Falsi Method

Q.1. Find the real root of the equation $x \log_{10} x - 1.2 = 0$ correct upto four decimal places.

Ans.: Given $f(x) = x \log_{10} x - 1.2$ _ _ _ (1)

In this method following formula is used -

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})\ f(x_n)}{(f(x_n) - f(x_{n-1}))} \quad \text{\_ \_ \_ (2)}$$

Taking x = 1 in eq.(1)

$f(1) = 1. \log_{10} 1 - 1.2$

$= -2$ (which is negative)

Taking x = 2 in eq.(1)

$f(2) = 2. \log_{10} 2 - 1.2$

$= -0.5979$ (which is negative)

Taking x = 3 in eq.(1)

$f(3) = 3. \log_{10} 3 - 1.2$

$= 0.2313$ (which is positive)

$\because \quad f(2).f(3) < 0$

So the root of the given equation lies between 2 and 3.

let $x_1 = 2$ and $x_2 = 3$

$\therefore \quad f(x_1) = f(2) = -0.5979$

And $\quad f(x_2) = f(3) = 0.2313$

Now we want to find x_3 so using eq.(2)

$$x_3 = x_2 - \frac{(x_2 - x_1)\ f(x_2)}{f(x_2) - f(x_1)}$$

$$= 3 - \frac{(3-2) \times (0.2313)}{0.2313 - (-0.5979)}$$

$$= 3 - \frac{0.2313}{0.8292}$$

$= 3 - 0.2789 = 2.7211$

$f(x_3) = f(2.7211)$

$= 2.7211 \log_{10} 2.7211 - 1.2$

$= -0.01701$ (which is negative)

$\because \quad f(x_2).f(x_3) < 0$

Now to find x_4 using equation (2)

$$x_4 = x_3 - \frac{(x_3 - x_2)\ f(x_3)}{f(x_3) - f(x_2)}$$

$$= 2.7211 - \frac{(2.7211 - 3) \times (-0.0170)}{(-0.0170 - 0.2313)}$$

$$= 2.7211 - \frac{0.004743}{0.2483}$$

$= 2.7211 + 0.01910 = 2.7402$

Now

$f(x_4) = f(2.7402)$

$= 2.7402 \log_{10} 2.7402 - 1.2$

$= -0.0003890$ (which is negative)

$\because \quad f(x_2).f(x_4) < 0$

Now to find x_5 using equation (2)

$$x_5 = x_4 - \frac{(x_4 - x_2)\ f(x_4)}{[f(x_4) - f(x_2)]}$$

$$= 2.7402 - \frac{(2.7402 - 3)}{(-0.0004762 - 0.2313)} \times (-0.0004762)$$

$$= 2.7402 + \frac{(-0.2598)(-0.0004762)}{0.2317}$$

$$= 2.7402 + \frac{(0.0001237)}{0.2317}$$

$= 2.7402 + 0.0005341 = 2.7406$

$f(x_5) = f(2.7406)$

$= 2.7406 \log_{10} 2.7406 - 1.2$

$= -0.0000402$ (which is negative)

$\because \quad f(x_2).f(x_5) < 0$

To find x_6 using equation (2)

$$x_6 = x_5 - \frac{(x_5 - x_2)\ f(x_5)}{f(x_5) - f(x_2)}$$

$$= 2.7406 + \frac{(2.7406 - 3) \times (-0.000040)}{(-0.00004) - (0.2313)}$$

$= 2.7406 + 0.000010 = 2.7406$

$\therefore$ The approximate root of the given equation is 2.7406 which is correct upto four decimals.

Q.2. Find the real root of the equation $x^3 - 2x - 5 = 0$ correct upto four decimal places.

Ans.: Given equation is

$$f(x) = x^3 - 2x - 5 \qquad \text{---} (1)$$

In this method following formula is used :-

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})\ f(x_n)}{[f(x_n) - f(x_{n-1})]} \qquad \text{---} (2)$$

Taking x = 1 in equation (1)

$f(1) = 1 - 2 - 5 = -6$ (which is negative)

Taking x = 2 in equation (1)

$f(2) = 8 - 4 - 5 = -1$ (which is negative)

Taking x = 3

$f(3) = 27 - 6 - 5 = 16$ (which is positive)

Since $f(2).f(3) < 0$

So the root of the given equation lies between 2 and 3.

Let $x_1 = 2$ and $x_2 = 3$

$f(x_1) = f(2) = -1$

and $f(x_2) = f(3) = 16$

Now to find x_3 using equation (2)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

$$= 3 - \frac{(3 - 2)}{16 + 1} \times 16$$

$$= 3 - \frac{16}{17} = 2.0588$$

$$f(x_3) = (2.0558)^3 - 2\,(2.0588) - 5$$

$$= 8.7265 - 4.1176 - 5$$

$$= -\,0.3911 \text{ (which is negative)}$$

$\therefore \quad f(x_2).f(x_3) < 0$

Now to find x_4 using equation (2)

$$x_4 = x_3 - \frac{(x_3 - x_2)}{[f(x_3) - f(x_2)]} \times f(x_3)$$

$$= 2.0588 - \frac{(2.0588 - 3)}{-0.3911 - 16} \times (-0.3911)$$

$$= 2.0588 + \frac{(-0.9412)\times(-0.3911)}{16.3911} = 2.0812$$

$\therefore \quad f(x_4) = 9.0144 - 4.1624 - 5$

$$= -\,0.148 \text{ (which is negative)}$$

So $\quad f(x_2)\,.\,f(x_4) < 0$

Now using equation (2) to find x_5

$$x_5 = x_4 - \frac{(x_4 - x_2)}{[f(x_4) - f(x_2)]} \times f(x_4)$$

$$= 2.0812 - \frac{(2.0812 - 3)}{(-0.148 - 16)} \times (-0.148)$$

$$= 2.0812 + \frac{(-0.9188)\times(-0.148)}{16.148}$$

$$= 2.0812 + 8.4210 \text{ x } \frac{(x_5 - x_2)\times f(x_5)}{f(x_5) - f(x_2)} 10^{-3}$$

$$= 2.0896$$

. $f(x_5) = 9.1240 - 4.1792 - 5$

$= -0.0552$ (which is negative)

$f(x_2).f(x_5) < 0$

Now using equation (2) to find x_6

$$x_6 = x_5 - \frac{(x_5 - x_2) \times f(x_5)}{f(x_5) - f(x_2)}$$

$$= 2.0896 - \frac{(2.0896 - 3)}{(-0.0552 - 16)} \times (-0.0552)$$

$$= 2.0896 + \frac{(0.05025)}{16.0552}$$

$$= 2.0927$$

$\therefore$ $f(x_6) = 9.1647 - 4.1854 - 5$

$= -0.0207$ (which is negative)

So $f(x_2).f(x_6) < 0$

Now using equation (2) to find x_7

$$x_7 = x_6 - \frac{(x_6 - x_2)}{f(x_6) - f(x_2)} \times f(x_6)$$

$$= 2.0927 - \frac{(2.0927 - 3)}{(-0.0207 - 16)} \times (-0.0207)$$

$$= 2.0927 + \frac{(-0.9073)\,(-0.0207)}{16.0207}$$

$$= 2.0927 + 1.1722 \text{ x } 10^{-3}$$

$$= 2.0938$$

Now $f(x_7) = 9.1792 - 4.1876 - 5$

$= -0.0084$ (which is negative)

So $f(x_2).f(x_7) < 0$

Now using equation (2) to find x_8

$$x_8 = x_7 - \frac{(x_7 - x_2)}{f(x_7) - f(x_2)} \times f(x_7)$$

$$= 2.0938 - \frac{(2.0938 - 3)}{(-0.0084 - 16)} \times (-0.0084)$$

$$= 2.0938 + \frac{(-0.9062)\ (-0.0084)}{16.0084}$$

$$= 2.0938 + 4.755 \times 10^{-4}$$

$$= 2.09427$$

∴ $f(x_8) = 9.1853 - 4.18854 - 5$

$= -\ 0.00324$ (which is negative)

So $f(x_2).f(x_8) < 0$

Now using equation (2) to find x_9

$$x_9 = x_8 - \frac{(x_8 - x_2)}{f(x_8) - f(x_2)} \times f(x_8)$$

$$= 2.09427 - \frac{(2.09427 - 3)}{(-0.00324 - 16)} \times (-0.00324)$$

$$= 2.09427 - \frac{(-0.90573)\ (-0.00324)}{16.00324}$$

$$= 2.0944$$

∴ The real root of the given equation is 2.094 which is correct upto three decimals.

□

Newton Raphson Method

Newton Rapson's method is a method for finding successively approximation to the roots.

Derivation of Newton Rapson's Method

We have the Taylor series expansion

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 .... + \frac{f^n(x_i)}{n!}h^n$$

Where $h = x_{i+1} - x_1$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_1) + \frac{f''(x_i)}{2!}(x_{i+1} - x_1)^2 .... + \frac{f^n(x_i)}{n!}(x_{i+1} - x_1)^n$$

Truncate the series

We get

$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i)$

At the intersection of the x axis $f(x_{i+1})$ =0 so we have

$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$

Solving this we get

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Q.1. Find the root of the equation $x^2 - 5x + 2 = 0$ correct upto 5 decimal places. (use Newton Raphson Method.)

Ans.: : Given $f(x) = x^2 - 5x + 2 = 0$

Taking x = 0

$f(0) = 2$ (which is positive)

Taking $x = 1$

$f(1) = 1 - 5 + 2 = -2$ (which is negative)

$f(0) \cdot f(1) < 0$

$\therefore$ The root of the given equation lies between 0 and 1

Taking initial approximation as

$$x_1 = \frac{0+1}{2} = 0.5$$

$f(x) = x^2 - 5x + 2$

$f'(x) = 2x - 5$

Since $x_1 = 0.5$

$$f(x_1) = (0.5)^2 - 5(0.5) + 2$$
$$= 0.25 - 2.5 + 2$$
$$= -0.25$$

$$f'(x_1) = 2(0.5) - 5$$
$$= 1 - 5$$
$$= -4$$

Now finding x_2

$$x_2 = 0.5 - \frac{(-0.25)}{-4}$$
$$= 0.5 - \frac{0.25}{4}$$
$$= 0.4375$$

$$f(x_2) = (0.4375)^2 - 5(0.4375) + 2$$
$$= 0.19140 - 2.1875 + 2$$
$$= 0.003906$$

$$f'(x_2) = 2(0.4375) - 5$$
$$= -4.125$$

Now finding x_3

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)}$$

$$= 0.4375 - \frac{0.003906}{(-4.125)}$$

$$= 0.4375 + 0.0009469$$

$$= 0.43844$$

$$f(x_3) = (0.43844)^2 - 5(0.43844) + 2$$

$$= 0.19222 - 2.1922 + 2$$

$$= 0.00002$$

$$f'(x_3) = 2 \times (0.43844) - 5$$

$$= -4.12312$$

$$x_4 = x_3 - \frac{f(x_3)}{f^1(x_3)}$$

$$= 0.43844 - \frac{0.00002}{(-4.12312)}$$

$$= 0.43844 + 0.00000485$$

$$= 0.43844$$

Hence the root of the given equation is 0.43844 which is correct upto five decimal places.

Q.2. Apply Newton Raphson Method to find the root of the equation 3x – cos x – 1 = 0 correct the result upto five decimal places.

Ans.: Given equation is

$$f(x) = 3x - \cos x - 1$$

Taking x = 0

$$f(0) = 3(0) - \cos 0 - 1$$

$$= -2$$

Now taking x = 1

$$f(1) = 3(1) - \cos(1) - 1$$

$$= 3 - 0.5403 - 1$$

$$= 1.4597$$

Taking initial approximation as

$$x_1 = \frac{0+1}{2} = 0.5$$

$f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

At $x_1 = 0.5$

$f(x_1) = 3\,(0.5) - \cos\,(0.5) - 1$

$= 1.5 - 0.8775 - 1$

$= -\,0.37758$

$f'(x_1) = 3 - \sin\,(0.5)$

$= 3.47942$

Now to find x_2 using following formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{(-0.37758)}{(3.47942)}$$

$= 0.5 + 0.10851$

$= 0.60852$

$f(x_2) = 3\,(0.60852) - \cos\,(0.60852) - 1$

$= 1.82556 - 0.820494 - 1$

$= 0.005066$

$f'(x_2) = 3 + \sin\,(0.60852)$

$= 3.57165$

Now finding x_3

$$x_3 = 0.60852 - \frac{(0.005066)}{(3.57165)}$$

$= 0.60852 - 0.0014183$

$= 0.60710$

$f(x_3) = 3\,(0.60710) - \cos\,(0.60710) - 1$

$= 1.8213 - 0.821305884 - 1$

$= -\,0.00000588$

$f'\,(x_3) = 3 + \sin\,(0.60710)$

$= 3 + 0.57048$

$= 3.5704$

Now to find x_4 using following formula

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.60710 - \frac{(-0.00000588)}{3.5704}$$

$$= 0.60710 + 0.00000164$$

$$= 0.60710$$

Which is same as x_3

Hence the root of the given equation is x = 0.60710 which is correct upto five decimal places.

□

Iterative Method

Q.1. Find a root of the equation $x^3 + x^2 - 1 = 0$ in the interval (0,1) with an accuracy of 10^{-4}.

Ans.: Given equation is $f(x) = x^3 + x^2 - 1 = 0$

Rewriting above equation in the form

$$x = \phi(x)$$

The given equation can be expressed in either of the form :

(i) $x^3 + x^2 - 1 = 0$

$$x^3 + x^2 = 1$$

$$x^2 (x + 1) = 1$$

$$x^2 = \frac{1}{1 + x}$$

$$x = \frac{1}{\sqrt{(1+x)}} \qquad \text{---} (1)$$

(ii) $x^3 + x^2 - 1 = 0$

$$x^2 = 1 - x^3$$

$$x = (1 + x^3)^{-(1/2)} \qquad \text{---} (2)$$

(iii) $x^3 + x^2 - 1 = 0$

$$x^3 = 1 - x^2$$

$$x = (1 - x^2)^{1/3} \qquad \text{---} (3)$$

Comparing equation (1) with x – g (x) = 0 we find that

$$g(x) = \frac{1}{\sqrt{(1+x)}}$$

$$g(x) = (1 + x)^{-1/2}$$

$$g'(x) = -\tfrac{1}{2} (1 + x)^{-3/2}$$

$$|g'(x)| = \tfrac{1}{2} (1 + x)^{3/2}$$

$$= \frac{1}{2(1+x)^{3/2}} < 1$$

Now comparing equation (2) with x – g(x) = 0

We find that $g(x) = (1 - x^3)^{1/2}$

$$g'(x) = \tfrac{1}{2} (1 + x^3)^{-1/2} \times (-3x^2)$$

$$= \frac{-3}{2} \frac{0+1}{2}$$

$$|g'(x)| = \frac{3}{2} \frac{x^2}{(1-x^2)^{1/2}}$$

Which is not less than one.

Now comparing equation (3) with x – g(x) = 0

$$g(x) = (1 - x^2)^{1/3}$$

$$g'(x) = \frac{1}{3} (1 - x^2)^{-2/3} \times (-2x)$$

$$= -\frac{2}{3} \frac{x}{(1-x^2)^{1/2}}$$

$$|g'(x)| = \frac{2}{3} \frac{x}{(1-x^2)^{2/3}}$$

Which is not less than one.

Hence this method is applicable only to equation (1) because it is convergent for all $x \in (0, 1)$

Now taking initial approximation

$$x_1 = \frac{0+1}{2} = 0.5$$

So $x_2 = \dfrac{1}{\sqrt{(1+x_1)}}$ [using iteration scheme $x_{n+1} = \dfrac{1}{\sqrt{(x_n+1)}}$]

$$x_2 = \frac{1}{\sqrt{0.5+1}} = \frac{1}{\sqrt{1.5}} = 0.81649$$

Similarly

$$x_3 = \frac{1}{\sqrt{(x_2+1)}} = \frac{1}{\sqrt{0.81649+1}} = 0.7419$$

$$x_4 = \frac{1}{\sqrt{(x_3+1)}} = \frac{1}{\sqrt{0.7419+1}} = 0.7576$$

$$x_5 = \frac{1}{\sqrt{(x_4+1)}} = \frac{1}{\sqrt{0.7576+1}} = 0.7542$$

$$x_6 = \frac{1}{\sqrt{(x_5+1)}} = \frac{1}{\sqrt{0.7542+1}} = 0.7550$$

$$x_7 = \frac{1}{\sqrt{(x_6+1)}} = \frac{1}{\sqrt{0.7550+1}} = 0.7548$$

$$x_8 = \frac{1}{\sqrt{(x_7+1)}} = \frac{1}{\sqrt{0.7548+1}} = 0.7548$$

Hence the approximate root of the given equation is x = 0.7548

Q.2. Find the root of the equation 2x = cos x + 3 correct upto 3 decimal places.

Ans.: Given equation is

$$f(x) = 2x - \cos x - 3 = 0$$

Rewriting above equation in the form x = g(x)

$$\Rightarrow 2x = \cos x + 3$$

$$\Rightarrow x = \frac{\cos x + 3}{2} \qquad --- (1)$$

Comparing above equation with the following equation x = g(x) we find the

$$g(x) = \frac{\cos x + 3}{2} = \frac{\cos x}{2} + \frac{3}{2}$$

$$g'(x) = \frac{-\sin x}{2}$$

$$|g'(x)| = \frac{\sin x}{2}$$

For $x \in (1, 2)$

$$|\sin x| < 1$$

Hence the iterative scheme $x_{n+1} = \frac{\cos(x_n) + 3}{2}$ is convergent.

Now taking initial approximation $x_1 = 1.5$

$$\therefore \quad x_2 = \frac{\cos x_1 + 3}{2} = \frac{\cos(1.5) + 3}{2} = 1.5353$$

$$x_3 = \frac{\cos(x_2) + 3}{2} = \frac{\cos(1.5353) + 3}{2} = 1.5177$$

$$x_4 = \frac{\cos(x_3) + 3}{2} = \frac{\cos(1.5177) + 3}{2} = 1.5265$$

$$x_5 = \frac{\cos(x_4) + 3}{2} = \frac{\cos(1.5265) + 3}{2} = 1.5221$$

$$x_6 = \frac{\cos(x_5) + 3}{2} = \frac{\cos(1.5221) + 3}{2} = 1.5243$$

$$x_7 = \frac{\cos(x_6) + 3}{2} = \frac{\cos(1.5243) + 3}{2} = 1.5230$$

$$x_8 = \frac{\cos(x_7) + 3}{2} = \frac{\cos(1.5230) + 3}{2} = 1.523$$

Which is same as x_7

Hence the root of the given equation is x = 1.523 (which is correct upto 3 decimals)

Q.3. Find the root of the equation $xe^x = 1$ in the internal (0, 1) (use iterative Method)

Ans.: Given equation is $xe^x - 1 = 0$

Rewriting above equation in the form of x = g (x)

$xe^x - 1 = 0$

$xe^x = 1$

$x = e^{-x}$

Comparing it with the equation x = g (x) we find that

$g(x) = e^{-x}$

$g'(x) = -e^{-x}$

$|g'(x)| = e^{-x} < 1$

Hence the iterative scheme is

$x_{n+1} = e^{-x_n}$

Now taking initial approximation

$x_1 = 0.5$

$x_2 = e^{-x_1} = e^{-(0.5)} = 0.60653$

$x_3 = e^{-x_2} = e^{-(0.6065)} = 0.5452$

$x_4 = e^{-x_3} = e^{-(0.5452)} = 0.5797$

$x_5 = e^{-x_4} = e^{-0.5797} = 0.5600$

$x_6 = e^{-x_5} = e^{-0.5600} = 0.5712$

$x_7 = e^{-x_6} = e^{-(0.5712)} = 0.5648$

$x_8 = e^{-x_7} = e^{-(0.5648)} = 0.5684$

$x_9 = e^{-x_8} = e^{-(0.5684)} = 0.5664$

$x_{10} = e^{-x_9} = e^{-(0.5664)} = 0.5675$

Now

$x_{11} = e^{-x_{10}} = e^{-0.5675} = 0.5669$

$x_{12} = e^{-x_{11}} = e^{-0.5669} = 0.5672$

$x_{13} = e^{-x_{12}} = e^{-(0.5672)} = 0.5671$

$x_{14} = e^{-x_{13}} = e^{-(0.5671)} = 0.5671$

Hence the approximate root the given equation is x = 0.5671

□ □ □

Gauss Elimination Method

Q.1. Use gauss elimination method to solve :

x + y + z = 7

3x + 3y + 4z = 24

2x + y + 3z = 16

Ans.: Since in the first column the largest element is 3 in the second equation, so interchanging the first equation with second equation and making 3 as first pivot.

$3x + 3y + 4z = 24$ _ _ _ (1)

$x + y + z = 7$ _ _ _ (2)

$2x + y + 3z = 16$ _ _ _ (3)

Now eliminating x form equation (2) and equation (3) using equation (1)

-3 × equation (2) + 2 × equation (1), 3 × equation (3) – 2 × equation (1)

we get

$$\begin{array}{r} -3x - 3y - 3z = -21 \\ 3x + 3y + 4z = 24 \\ \hline z = 3 \end{array} \quad \text{and} \quad \begin{array}{r} 6x + 3y + 9z = 48 \\ -6x \pm 6y \pm 8z = 48 \\ \hline -3y + z = 0 \\ = 3y - z = 0 \end{array}$$

$3x + 3y + 4z = 24$ _ _ _ (4)

$z = 3$ _ _ _ (5)

$3y - z = 0$ _ _ _ (6)

Now since the second row cannot be used as the pivot row since $a_{22} = 0$ so interchanging the equation (5) and (6) we get

$3x + 3y + 4z = 24$ _ _ _ (7)

$3y - z = 0$ _ _ _ (8)

$z = 3$ _ _ _ (9)

Now it is upper triangular matrix system. So by back substitution we obtain.

$\boxed{z = 3}$

From equation (8)

$3y - 3 = 0$

$3y = 3$

$\boxed{y = 1}$

From equation (7)

$3x + 3(1) + 4(3) = 24$

$3x + 3 + 12 = 24$

$3x + 15 = 24$

$3x = 9$

$\boxed{x = 3}$

Hence the solution fo given system of linear equation is

$x = 3 \quad , \quad y = 1 \quad , \quad z = 3$

Q.2. Solve the following system of linear equation by Gauss Elimination Method :

$\mathbf{2x_1 + 4x_2 + x_3 = 3}$

$\mathbf{3x_1 + 2x_2 - 2x_3 = -2}$

$\mathbf{x_1 - x_2 + x_3 = 6}$

Ans.: Since in the first column the largest element is 3 in the second row, so interchanging first equation with second equation and making 3 as first pivot.

$3x_1 + 2x_2 - 2x_3 = -2 \qquad \_\_\_ (1)$

$2x_1 + 4x_2 + x_3 = 3 \qquad \_\_\_ (2)$

$x_1 - x_2 + x_3 = 6 \qquad \_\_\_ (3)$

Eliminating x_1 form equation (2) and equation (3) using equation (1)

-3 × equation (2) + 2 × equation (1) and + 3 × equation (3) – equation (1)

$$\begin{array}{r} -6x_1 - 12x_2 - 3x_3 = -9 \\ 6x_1 + 4x_2 - 4x_3 = -4 \\ \hline -8x_2 - 7x_3 = -13 \end{array} \quad \text{and} \quad \begin{array}{r} 3x_1 - 3x_2 + 3x_3 = 18 \\ 3x_1 + 2x_2 - 2x_3 = -2 \\ - \quad - \quad + \quad + \\ \hline -5x_2 + 5x_3 = 20 \end{array} \quad +$$

$8x_2 + 7x_3 = 13 \qquad\qquad x_2 - x_3 = -4$

So the system now becomes :

$3x_1 + 2x_2 - 2x_3 = -2 \qquad \_\_\_ (4)$

$8x_2 + 7x_3 = 13 \qquad \_\_\_ (5)$

$x_2 - x_3 = -4$ ——— (6)

Now eliminating x_2 from equation (6) using equation (5) {8 × equation (6) – equation (5)}

$$
\begin{array}{r}
8x_2 - 8x_3 = -32 \\
-\ \underline{8x_2 + 7x_3 = -13} \\
-15x_3 = -45
\end{array}
$$

$x_3 = 3$

So the system of linear equation is

$3x_1 + 2x_2 - 2x_3 = -2$ ——— (7)

$8x_2 + 7x_3 = 13$ ——— (8)

$x_3 = 3$ ——— (6)

Now it is upper triangular system so by back substitution we obtain

$x_3 = 3$

From equation (8)

$8x_2 + 7(3) = 13$

$8x_2 = 13 - 21$

$8x_2 = -8$

$x_2 = -1$

From equation (9)

$3x_1 + 2(-1) - 2(3) = -2$

$3x_1 = -2 + 2 + 6$

$3x_1 = 6$

$x_1 = 2$

∴ Hence the solution of the given system of linear equation is :

$x_1 = 2$, $x_2 = -1$, $x_3 = 3$

❐

Gauss-Jordan Elimination Method

Q.1. Solve the following system of equations :

$$10x_1 + 2x_2 + x_3 = 9 \qquad \_\,\_\,\_ \ (1)$$

$$2x_1 + 20x_2 - 2x_3 = -44 \qquad \_\,\_\,\_ \ (2)$$

$$-2x_1 + 3x_2 + 10x_3 = 22 \qquad \_\,\_\,\_ \ (3)$$

Use Gauss Jordan Method.

Ans.: Since in the given system pivoting is not necessary. Eliminating x_1 from equation (2) and equation (3) using equation (1)

5 × equation (2) – equation (1) , 5 × equation (3) **+** equation (1)

$$\begin{array}{r} 10x_1 - 100x_2 - 10x_3 = -220 \\ -10x_1 \mp 2x_2 \mp x_3 = -9 \\ \hline 98x_2 - 11x_3 = -229 \end{array} \quad \text{and} \quad \begin{array}{r} -10x_1 + 15x_2 + 50x_3 = 110 \\ 10x_1 + 2x_2 + x_3 = 9 \\ \hline 17x_2 + 51x_3 = 119 \\ = x_2 + 3x_3 = 7 \end{array}$$

Now the system of equation becomes

$$10x_1 + 2x_2 + x_3 = 9 \qquad \_\,\_\,\_ \ (4)$$

$$98x_2 - 11x_3 = -229 \qquad \_\,\_\,\_ \ (5)$$

$$x_2 + 3x_3 = 7 \qquad \_\,\_\,\_ \ (6)$$

Now eliminating x_2 from equation (4) and (6) using equation (5)

98 × equation (6) – equation (5) , 49 × equation (4) **-** equation (5)

$$\begin{array}{r} 98x_2 + 294x_3 = 686 \\ -98x_2 \mp 11x_3 = \mp 229 \\ \hline 305x_3 = 915 \\ x_3 = 3 \end{array} \qquad \begin{array}{r} 490x_1 + 98x_2 + 49x_3 = 441 \\ -98x_2 \pm 11x_3 = \mp 9 \\ \hline 490x_1 + 60x_3 = 670 \\ = 49x_1 + 6x_3 = 67 \end{array}$$

Now the system of equation becomes :

$$49x_1 + 0 + 6x_3 = 67 \qquad \_\,\_\,\_ \ (7)$$

$98x_2 - 11x_3 = -229$ _ _ _ (8)

$x_3 = 3$ _ _ _ (9)

Hence it reduces to upper triangular system now by back substitution.

$x_3 = 3$

From equation (8)

$98x_2 - 11 \times 3 = -229$

$98x_2 = -229 + 33$

$98x_2 = -196$

$x_2 = -2$

From equation (7)

$49x_1 + 6(3) = 67$

$49x_1 = 67 - 18$

$49x_1 = 49$

$x_1 = 1$

Thus the solution of the given system of linear equation is

$x_1 = 1$, $x_2 = -2$, $x_3 = 3$

Jacobi Method

Q.1. Solve the following system of equation by Jacobi Method.

$$\mathbf{83x_1 + 11x_2 - 4x_3 = 95}$$

$$\mathbf{7x_1 + 52x_2 + 13x_3 = 104}$$

$$\mathbf{3x_1 + 8x_2 + 29x_3 = 71}$$

Ans.: Since the given system of equation is

$$83x_1 + 11x_2 - 4x_3 = 95 \qquad \_\_\_ (1)$$

$$7x_1 + 52x_2 + 13x_3 = 104 \qquad \_\_\_ (2)$$

$$3x_1 + 8x_2 + 29x_3 = 71 \qquad \_\_\_ (3)$$

The diagonal elements in the given system of linear equations is not zero so the equation (1), (2) and (3) can be written as :

$$x_1^{(n+1)} = \frac{1}{83}[95 - 11\,x_2^{(n)} + 4\,x_3^{(n)}]$$

$$x_2^{(n+1)} = \frac{1}{52}[104 - 7\,x_1^{(n)} - 13\,x_3^{(n)}] \text{ and}$$

$$x_3^{(n+1)} = \frac{1}{29}[71 - 3\,x_1^{(n)} - 8\,x_2^{(n)}]$$

Now taking initial approximation as :

$$x_1^{(0)} = 0 \quad ; \quad x_2^{(0)} = 0 \quad \text{and} \quad x_3^{(0)} = 0$$

Now for first approximation :

$$x_1^{(1)} = \frac{1}{83}[95 - 11\,x_2^{(0)} + 4\,x_3^{(0)}] = 1.1446$$

$$x_2^{(1)} = \frac{1}{52}[104 - 7\,x_1^{(0)} - 13\,x_3^{(0)}] = 2$$

$$x_3^{(1)} = \frac{1}{29}[71 - 3\,x_1^{(0)} - 8\,x_2^{(0)}] = 2.4483$$

Similarly second approximation :

$$x_1^{(2)} = \frac{1}{83}[95 - 11\,x_2^{(1)} + 4\,x_3^{(1)}]$$

$$= \frac{1}{83}[95 - 11(2) + 4(2.4483)] = 0.9975$$

$$x_2^{(2)} = \frac{1}{52}[104 - 7\,x_1^{(1)} - 13\,x_3^{(1)}]$$

$$= \frac{1}{52}[104 - 7(1.1446) - 13(2.4483)] = 1.2338$$

$$x_3^{(2)} = \frac{1}{29}[71 - 3\,x_1^{(1)} - 8\,x_2^{(1)}]$$

$$= \frac{1}{29}[71 - 3(1.1446) - 8 \times 2] = 1.7781$$

Now the third iteration :

$$x_1^{(3)} = \frac{1}{83}[95 - 11\,x_2^{(2)} + 4\,x_3^{(2)}]$$

$$= \frac{1}{83}[95 - 11\times (1.2338) + 4(1.7781)] = 1.0668$$

$$x_2^{(3)} = \frac{1}{52}[104 - 7\,x_1^{(2)} - 13\,x_3^{(2)}]$$

$$= \frac{1}{52}[104 - 7 \times (0.9975) - 13 \times (1.7781)] = \frac{1}{52}[73.9022]$$

$$= 1.4212$$

$$x_3^{(3)} = \frac{1}{29}[71 - 3\,x_1^{(2)} - 8\,x_2^{(2)}]$$

$$= \frac{1}{29}[71 - 3 \times (0.9975) - 8 \times (1.2338)] = 2.0047$$

Similarly other iterations are :

$$x_1^{(4)} = 1.0528$$

$$x_2^{(4)} = 1.3552$$

$$x_3^{(4)} = 1.9459$$

$$x_1^{(5)} = 1.0588$$

$$x_2^{(5)} = 1.3718$$

$x_3^{(5)} = 1.9655$

$x_1^{(6)} = 1.0575$

$x_2^{(6)} = 1.3661$

$x_3^{(6)} = 1.9603$

$x_1^{(7)} = 1.0580$

$x_2^{(7)} = 1.3676$

$x_3^{(7)} = 1.9620$

$x_1^{(8)} = 1.0579$

$x_2^{(8)} = 1.3671$

$x_3^{(8)} = 1.9616$

$x_1^{(9)} = 1.0579$

$x_2^{(9)} = 1.3671$

$x_3^{(9)} = 1.9616$

Thus the values obtained by successive iteration is given by following table :

x	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n+1)}$	$x_2^{(n+1)}$	$x_3^{(n+1)}$
0	0	0	0	1.1446	2	2.4483
1	1.1446	2	2.4483	0.9975	1.2338	1.7781
2	0.9975	1.2338	1.7781	1.0667	1.4211	2.0047
3	1.0667	1.4211	2.0047	1.0528	1.3552	1.9459
4	1.0528	1.3552	1.9459	1.0587	1.3718	1.9655
5	1.0587	1.3718	1.9655	1.0575	1.3661	1.9603
6	1.0575	1.3661	1.9603	1.0580	1.3676	1.9620
7	1.0580	1.3676	1.9620	1.0579	1.3671	1.9616
8	1.0579	1.3671	1.9616	1.0579	1.3671	1.9616

Thus the solution is

$x_1 = 1.0579$; $x_2 = 1.3671$ and $x_3 = 1.9616$

□ □ □

Gauss Seidel Method

[This method is also called the method of successive displacement]

Q.1. Solve the following linear equation :

$2x_1 - x_2 + x_3 = 5$

$x_1 + 2x_2 + 3x_3 = 10$

$x_1 + 3x_2 - 2x_3 = 7$

(Use Gauss Seidel Method)

Ans.: Above system of equations can be written as :

$$2x_1 - x_2 + x_3 = 5 \qquad \text{--- (1)}$$

$$x_1 + 3x_2 - 2x_3 = 7 \qquad \text{--- (2)}$$

$$x_1 + 2x_2 + 3x_3 = 10 \qquad \text{--- (3)}$$

Iterative equations are :

$$x_1^{(n+1)} = \frac{1}{2}[5 + x_2^{(n)} - x_3^{(n)}] \qquad \text{--- (4)}$$

$$x_2^{(n+1)} = \frac{1}{3}[7 - x_1^{(n+1)} + 2x_3^{(n)}] \qquad \text{--- (5)}$$

$$x_3^{(n+1)} = \frac{1}{3}[10 - x_1^{(n+1)} - 2x_2^{(n+1)}] \qquad \text{--- (6)}$$

Taking initial approximations as :

$x_1^{(0)} = 0$; $x_2^{(0)} = 0$ and $x_3^{(0)} = 0$

First approximation is :

$$x_1^{(1)} = \frac{1}{2}[5 + x_2^{(0)} - x_3^{(0)}]$$

$$= \frac{1}{2}[5 + 0 - 0] = \frac{5}{2} = 2.5$$

$$x_2^{(1)} = \frac{1}{3}[7 - x_1^{(1)} + 2\, x_3^{(0)}]$$

$$= \frac{1}{3}[7 - 2.5 + 2 \times 0] = \frac{1}{3}(4.5) = 1.5$$

$$x_3^{(1)} = \frac{1}{3}\ [10 - x_1^{(1)} - 2\, x_2^{(1)}]$$

$$= \frac{1}{3}\ [10 - 2.5 - 2 \times 1.5] = 1.5$$

Now second approximation :

$$x_1^{(2)} = \frac{1}{2}[5 + x_2^{(1)} - x_3^{(1)}]$$

$$= \frac{1}{2}[5 + (1.5) - 1.5] = 2.5$$

$$x_2^{(2)} = \frac{1}{3}[7 - x_1^{(2)} + 2\, x_3^{(1)}]$$

$$= \frac{1}{3}[7 - 2.5 + 2\ (1.5)] = 2.5$$

$$x_3^{(2)} = \frac{1}{3}\ [10 - x_1^{(2)} - 2\, x_2^{(2)}]$$

$$= \frac{1}{3}\ [10 - 2.5 - 2 \times 2.5] = 0.8333$$

$$x_1^{(3)} = \frac{1}{2}[5 + x_2^{(2)} - x_3^{(2)}]$$

$$= \frac{1}{2}[5 + 2.5 - 0.8333] = 3.3333$$

$$x_2^{(3)} = \frac{1}{3}[7 - x_1^{(3)} + 2\, x_3^{(2)}]$$

$$= \frac{1}{3}[7 - 3.3333 + 2 \times 0.8333] = 1.7777$$

$$x_3^{(3)} = \frac{1}{3}\ [10 - x_1^{(3)} - 2\, x_2^{(3)}]$$

$$= \frac{1}{3}\ [10 - 3.3333 - 2 \times 1.7777] = 1.0371$$

$\therefore \quad x_1^{(3)} = 3.3333 \quad , \quad x_2^{(3)} = 1.7777 \quad , \quad x_3^{(3)} = 1.0371$

$$x_1^{(4)} = \frac{1}{2}[5 + x_2^{(3)} - x_3^{(3)}]$$

$$= \frac{1}{2}[5 + 1.7777 - 1.0371] = 2.8703$$

$x_2^{(4)} = 2.0679$

$x_3^{(4)} = 0.9980$

$\therefore \quad x_1^{(4)} = 2.8703 \quad , \quad x_2^{(4)} = 2.0679 \quad , \quad x_3^{(4)} = 0.9980$

Now $x_1^{(5)} = 3.035$

$x_2^{(5)} = 1.9870$

$x_3^{(5)} = 0.9970$

$x_1^{(6)} = 2.9950$

$x_2^{(6)} = 1.9997$

$x_3^{(6)} = 1.0019$

$x_1^{(7)} = 2.9989$

$x_2^{(7)} = 2.0016$

$x_3^{(7)} = 0.9993$

$x_1^{(8)} = 3.0011$

$x_2^{(8)} = 1.9991$

$x_3^{(8)} = 1.0002$

$x_1^{(9)} = 2.9994$

$x_2^{(9)} = 2.0003$

$x_3^{(9)} = 1$

$x_1^{(10)} = 3.0001$

$x_2^{(10)} = 1.9999$

$x_3^{(10)} = 1$

$x_1^{(11)} = 2.9999$

$x_2^{(11)} = 2$

$x_3^{(11)} = 1$

$x_1^{(12)} = 3$

$x_2^{(12)} = 2$

$x_3^{(12)} = 1$

$x_1^{(13)} = 3$

$x_2^{(13)} = 2$

$x_3^{(13)} = 1$

Hence the solution of the given system of linear equation is :

$x_1 = 3$, $x_2 = 2$, $x_3 = 1$

□ □ □

Picards Method

Q.1 Explain Picard's method to solve O.D.E with I.V.P?

Consider the differential equation

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0 \tag{3}$$

Integrating eqn. (3) between the limits x_0 and x and the corresponding limits y_0 and y, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$\Rightarrow \qquad y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

or,
$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \tag{4}$$

In equation (4), the unknown function y appears under the integral sign. This type of equation is called integral equation.

This equation can be solved by the method of successive approximations or iterations.

To obtain the first approximation, we replace y by y_0 in the R.H.S. of eqn. (4).

Now, the first approximation is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

The integrand is a function of x alone and can be integrated.

For a second approximation, replace y_0 by $y^{(1)}$ in $f(x, y_0)$ which gives

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\{x, y^{(1)}\}\,dx$$

Proceeding in this way, we obtain $y^{(3)}, y^{(4)}, \ldots\ldots, y^{(n-1)}$ and $y^{(n)}$ where

$$y^{(n)} = y_0 + \int_{x_0}^{x} f\{x, y^{(n-1)}\}\,dx \text{ with } y(x_0) = y_0$$

As a matter of fact, the process is stopped when the two values of y *viz.* $y^{(n-1)}$ and $y^{(n)}$ are the same to the desired degree of accuracy.

Q.2

Given the differential eqn. $\dfrac{dy}{dx} = \dfrac{x^2}{y^2 + 1}$

with the initial condition $y = 0$ when $x = 0$. Use Picard's method to obtain y for $x = 0.25$, 0.5 and 1.0 correct to three decimal places.

Ans

(*a*) The given initial value problem is

$$\frac{dy}{dx} = f(x, y) = \frac{x^2}{y^2 + 1}$$

where $y = y_0 = 0$ at $x = x_0 = 0$

We have first approximation,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 0 + \int_0^x \frac{x^2}{0+1}\, dx = \frac{1}{3}x^3 \qquad (5)$$

Second approximation,

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\{x, y^{(1)}\}\, dx$$

$$= 0 + \int_0^x \frac{x^2}{\left(\frac{x^3}{3}\right)^2 + 1}\, dx$$

$$= \left[\tan^{-1} \frac{x^3}{3}\right]_0^x = \tan^{-1} \frac{x^3}{3}$$

$$= \frac{1}{3}x^3 - \frac{1}{3}\left(\frac{1}{3}x^3\right)^3 + \ldots\ldots$$

$$= \frac{1}{3}x^3 - \frac{1}{81}x^9 + \ldots\ldots \qquad (6)$$

From (5) and (6), we see that $y^{(1)}$ and $y^{(2)}$ agree to the first term $\frac{x^3}{3}$. To find the range of values of x so that the series with the term $\frac{1}{3}x^3$ alone will give the result correct to three decimal places, we put

$$\frac{1}{81}x^9 \leq .0005$$

which gives, $\quad x^9 \leq .0405 \quad \text{or} \quad x \leq 0.7$

Hence, $\quad y(.25) = \frac{1}{3}(.25)^3 = .005$

and $\quad y(0.5) = \frac{1}{3}(0.5)^3 = .042$

To find $y(1.0)$, we make use of eqn. (6) which gives,

$$y(1.0) = \frac{1}{3} - \frac{1}{81} = 0.321.$$

Q.3

Use Picard's method to obtain y for x = 0.1. Given that:

$$\frac{dy}{dx} = 3x + y^2;\ y = 1 \text{ at } x = 0.$$

Ans

$f(x, y) = 3x + y^2$, $x_0 = 0$, $y_0 = 1$

First approximation, $y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx$

$$= 1 + \int_0^x (3x + 1)\, dx$$

$$= 1 + x + \frac{3}{2} x^2$$

Second approximation, $y^{(2)} = 1 + x + \frac{5}{2}x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 + \frac{9}{20}x^5$

Third approximation, $y^{(3)} = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5$

$$+ \frac{68}{45}x^6 + \frac{1157}{1260}x^7 + \frac{17}{32}x^8 + \frac{47}{240}x^9$$

$$+ \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

when $x = 0.1$, we have

$$y^{(1)} = 1.115, \quad y^{(2)} = 1.1264, \quad y^{(3)} = 1.12721$$

Thus, $y = 1.127$ when $x = 0.1$.

Q.4

Solve by Picard's method, the differential equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = x^3 (y + z)$$

where $y = 1$, $z = \frac{1}{2}$ *at* $x = 0$. *Obtain the values of* y *and* z *from III approximation when* $x = 0.2$ *and* $x = 0.5$.

Ans

Let $\phi(x, y, z) = z, \quad f(x, y, z) = x^3(y + z)$

Here $x_0 = 0, \quad y_0 = 1, \quad z_0 = \frac{1}{2}$

First approximation,

$$y^{(1)} = y_0 + \int_0^x \phi(x, y_0, z_0)\, dx = 1 + \int_0^x z_0\, dx$$

$$= 1 + \frac{1}{2} x$$

$$z^{(1)} = z_0 + \int_0^x f(x, y_0, z_0)\, dx = \frac{1}{2} + \int_0^x x^3 (y_0 + z_0)\, dx$$

$$= \frac{1}{2} + \frac{3}{2} \frac{x^4}{4}.$$

Second approximation,

$$y^{(2)} = 1 + \int_0^x z^{(1)}\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8} x^4 \right) dx$$

$$= 1 + \frac{x}{2} + \frac{3}{40}x^5$$

$$z^{(2)} = \frac{1}{2} + \int_0^x x^3\{y^{(1)} + z^{(1)}\}\,dx$$

$$= \frac{1}{2} + \int_0^x x^3\left(\frac{3}{2} + \frac{x}{2} + \frac{3}{8}x^4\right)dx$$

$$= \frac{1}{2} + \frac{3}{8}x^4 + \frac{x^5}{10} + \frac{3}{64}x^8$$

Third approximation,

$$y^{(3)} = 1 + \int_0^x z^{(2)}\,dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right)dx$$

$$= 1 + \frac{x}{2} + \frac{3}{40}x^5 + \frac{x^6}{60} + \frac{3x^9}{576}$$

$$z^{(3)} = \frac{1}{2} + \int_0^x x^3\{y^{(2)} + z^{(2)}\}\,dx$$

$$= \frac{1}{2} + \int_0^x x^3\left\{\frac{3}{2} + \frac{x}{2} + \frac{3}{8}x^4 + \frac{7}{40}x^5 + \frac{3}{64}x^8\right\}dx$$

$$= \frac{1}{2} + \frac{3}{2}.\frac{x^4}{4} + \frac{1}{2}.\frac{x^5}{5} + \frac{3}{8}.\frac{x^8}{8} + \frac{7}{40}.\frac{x^9}{9} + \frac{3}{64}.\frac{x^{12}}{12}$$

$$= \frac{1}{2} + \frac{3}{8}x^4 + \frac{x^5}{10} + \frac{3}{64}x^8 + \frac{7}{360}x^9 + \frac{3}{768}x^{12}$$

when $x = 0.2$

$$y^{(3)} = 1 + 0.1 + \frac{3}{40}(0.2)^5 + \frac{(0.2)^6}{60} + \frac{3}{576}\ (0.2)^9$$

$$= 1.100024 \text{ (leaving higher terms)}$$

$$z^{(3)} = \frac{1}{2} + \frac{3}{8}(.2)^4 + \frac{(.2)^5}{10} + \frac{3}{64}(.2)^8 + \frac{7}{360}(.2)^9 + \frac{3}{768}\ (.2)^{12}$$

$$= .500632 \text{ (leaving higher terms)}$$

when $x = 0.5$

$$y^{(3)} = 1 + \frac{.5}{2} + \frac{3}{40}(.5)^5 + \frac{(.5)^6}{60} + \frac{3}{576}\ (.5)^9$$

$$= 1.25234375$$

$$z^{(3)} = \frac{1}{2} + \frac{3}{8}(.5)^4 + \frac{(.5)^5}{10} + \frac{3}{64}\ (.5)^8 + \frac{7}{360}(.5)^9 + \frac{3}{768}(.5)^{12}$$

$$= .5234375.$$

Numerical Solution for Differential Equations [Euler's Method]

EULER'S METHOD

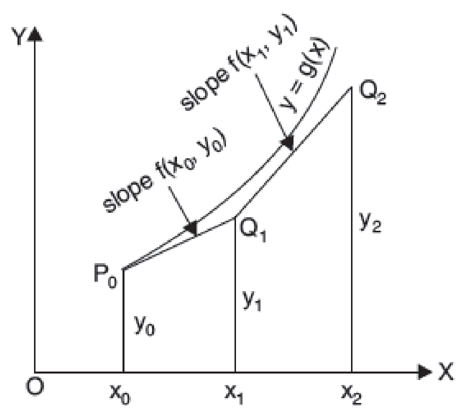
Euler's method is the simplest one-step method and has a limited application because of its low accuracy. This method yields solution of an ordinary differential equation in the form of a set of tabulated values.

In this method, we determine the change Δy is y corresponding to small increase in the argument x. Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{7}$$

Let $y = g(x)$ be the solution of (7). Let $x_0, x_1, x_2, ...$ be equidistant values of x.

In this method, we use the property that in a small interval, a curve is nearly a straight line. Thus at the point (x_0, y_0), we approximate the curve by the tangent at the point (x_0, y_0).



The eqn. of the tangent at $P_0(x_0, y_0)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{P_0} (x - x_0) = f(x_0, y_0)\,(x - x_0)$$

$$\Rightarrow \qquad y = y_0 + (x - x_0)\, f(x_0, y_0) \tag{8}$$

This gives the y-coordinate of any point on the tangent. Since the curve is approximated by the tangent in the interval (x_0, x_1), the value of y on the curve corresponding to $x = x_1$ is given by the above value of y in eqn. (8) approximately.

Putting $x = x_1 (= x_0 + h)$ in eqn. (8), we get

$$y_1 = y_0 + hf(x_0, y_0)$$

Thus Q_1 is (x_1, y_1)

Similarly, approximating the curve in the next interval (x_1, x_2) by a line through $Q_1(x_1, y_1)$ with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + hf(x_1, y_1)$$

In general, it can be shown that,

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

This is called Euler's Formula.

Q.1. Use Euler's Method to determine an approximate value of y at x = 0.2 from initial value problem $\frac{dx}{dy}$ = 1 – x + 4y y(0) = 1 taking the step size h = 0.1.

Ans.: Here h = 0.1, n = 2, $x_0 = 0$, $y_0 = 1$

Given $\frac{dx}{dy} = 1 - x + 4y$

Hence $y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.1\,[1 - x_0 + 4y_0]$

$= 1 + 0.1\,[1 - 0 + 4 \times 1]$

$= 1 + 0.1\,[1 + 4]$

$= 1 + 0.5 \times 5$

$= 1.5$

Similarly $y_2 = y_1 + hf(x_0 + h, y_1)$

$= 1.5 + 0.1[1 - 0.1 + 4 \times 1.5]$

$= 2.19$

Q.2. Using Euler's Method with step-size 0.1 find the value of y(0.5) from the following differential equation $\frac{dx}{dy} = x^2 + y^2$, y(0) = 0

Ans.: Here $h = 0.1$, $n = 5$, $x_0 = 0$, $y_0 = 0$ and $f(x, y) = x^2 + y^2$

Hence $y_1 = y_0 + hf(x_0, y_0)$

$= 0 + (0.1)\,[0^2 + 0^2]$

$= 0$

Similarly $y_2 = y_1 + hf(x_0 + h, y_1)$

$= 0 + (0.1)\,[(0.1)^2 + 0^2]$

$= (0.1)^3$

$= 0.001$

$y_3 = y_2 + hf[x_0 + 2h, y_2]$

$= 0.001 + (0.1)\,[(0.2)^2 + (0.001)^2]$

$= 0.001 + 0.1\,[0.04 + 0.00001]$

$= 0.001 + 0.1\,[0.040001]$

$= 0.005$

$y_4 = y_3 + hf[x_0 + 3h, y_3]$

$= 0.005 + (0.1)\,[(0.3)^2 + (0.005)^2]$

$= 0.005 + (0.1)\,[0.09 + 0.000025]$

$= 0.014$

$y_5 = y_4 + hf[x_0 + 4h, y_4]$

$= 0.014 + (0.1)\,[(0.4)^2 + (0.014)^2]$

$= 0.014 + (0.1)\,[0.16 + 0.00196]$

$= 0.031$

Hence the required solution is 0.031

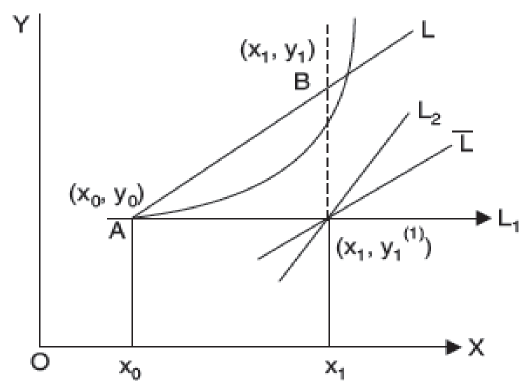
□ □ □

Numerical Solution for Differential Equations [Euler's Modified Method]

MODIFIED EULER'S METHOD

The modified Euler's method gives greater improvement in accuracy over the original Euler's method. Here the core idea is that we use a line through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$ where $y_1^{(1)} = y_0 + hf(x_0, y_0)$. This line approximates the curve in the interval (x_0, x_1).

Geometrically, if L_1 is the tangent at (x_0, y_0), L_2 is a line through $(x_1, y_1^{(1)})$ of slope $f(x_1, y_1^{(1)})$ and $\bar{L}$ is the line through $(x_1, y_1^{(1)})$ but with a slope equal to the average of $f(x_0, y_0)$ and $f(x_1, y_1^{(1)})$ then the line L through (x_0, y_0) and parallel to $\bar{L}$ is used to approximate the curve in the interval (x_0, x_1). Thus the ordinate of the point B will give the value of y_1. Now, the eqn. of the line AL is given by



$$y_1 = y_0 + (x_1 - x_0)\left[\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right]$$

$$= y_0 + h\left[\frac{f(x_0, y_0) + f(x_1, y_1^{(1)})}{2}\right]$$

A generalised form of Euler's modified formula is

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f\{x_1, y_1^{(n)}\}]\ ;\ n = 0, 1, 2, \ldots\ldots$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

The above iteration formula can be started by choosing $y_1^{(1)}$ from Euler's formula

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

Since this formula attempts to correct the values of y_{n+1} using the predicted value of y_{n+1} (by Euler's method), it is classified as a one-step predictor-corrector method.

Q.1. Using Euler's modified method, obtain a solution of the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with initial conditions y = 1 at x = 0 for the range $0 \le x \le 0.6$ in the step of 0.2. Correct upto four place of decimals.

Ans.: Here $f(x, y) = x + |\sqrt{y}|$

$x_0 = 0$, $y_0 = 1$, h = 0.2 and $x_n = x_0 + nh$

(i) At x = 0.2

First approximate value of y_1

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.2)\,[0 + 1]$$

$$= 1.2$$

Second approximate value of y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}\,[(0 + 1) + \{0.2 + \sqrt{1.2}\,\}]$$

$$= 1.2295$$

Third approximate value of y_1

$$y_1^{(3)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(2)})\}$$

$= 1 + \frac{0.2}{2}\ [(0 + 1) + \{0.2 + \sqrt{1.2295}\ \}]$

$= 1 + 0.1\ [1 + 1.30882821]$

$= 1.2309$

Fourth approximate value of y_1

$y_1^{(4)} = y_0 + \frac{h}{2}\ \{f(x_0, y_0) + f(x_1, y_1^{(3)})\}$

$= 1 + \frac{0.2}{2}\ [(0 + 1) + (0.2 + \sqrt{1.2309})]$

$= 1 + 0.1\ [1 + 1.30945]$

$= 1.2309$

Since the value of $y_1^{(3)}$ and $y_1^{(4)}$ is same

Hence at $x_1 = 0.2$, $y_1 = 1.2309$

(ii) At $x = 0.4$

First approximate value of y_2

$y_2^{(1)} = y_1 + hf(x_1, y_1)$

$= 1.2309 + (0.2)\ \{0.2 + \sqrt{1.2309}\}$

$= 1.4927$

Second approximate value of y_2

$y_2^{(2)} = y_1 + \frac{h}{2}\ [f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 1.2309 + \frac{0.2}{2}\ [(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.4927}\ )]$

$= 1.2309 + 0.1\ [1.309459328 + (1.621761024]$

$= 1.5240$

Third approximate value of y_2

$y_2^{(3)} = y_1 + \frac{h}{2}\ [f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 1.2309 + \frac{0.2}{2}\ [(1.309459328 + (0.4 + \sqrt{1.5240}\,)]$

$= 1.2309 + 0.1\ [1.309459328 + 1.634503949]$

$= 1.5253$

Fourth approximate value of y_2

$$y_2^{(4)} = y_1 + \frac{h}{2}\{f(x_1, y_1) + f(x_2, y_2^{(3)})\}$$

$$= 1.2309 + \frac{0.2}{2}[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.5253})]$$

$$= 1.2309 + 0.1\{1.309459328 + 1.635030364]$$

$$= 1.5253$$

Hence at $x = 0.4$, $y_2 = 1.5253$

(ii) At $x = 0.6$

First approximate value of y_3

$$y_3^{(1)} = y_2 + hf(x_2, y_2)$$

$$= 1.5253 + 0.2[0.4 + \sqrt{1.5253}]$$

$$= 1.8523$$

Second approximate value of y_3

$$y_3^{(2)} = y_2 + \frac{h}{2}\{f(x_2, y_2) + f(x_3, y_3^{(1)})\}$$

$$= 1.5253 + \frac{0.2}{2}[(0.4 + \sqrt{1.5253}) + (0.6+ \sqrt{1.8523})]$$

$$= 1.8849$$

Third approximate value of y_3

$$y_3^{(3)} = y_2 + \frac{h}{2}\{f(x_2, y_2) + f(x_3, y_3^{(2)})\}$$

$$= 1.5253 + \frac{0.2}{2}[(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8849})]$$

$$= 1.8851$$

Fourth approximate value of y_3

$$y_3^{(4)} = y_2 + \frac{h}{2}\{f(x_2, y_2) + f(x_3, y_3^{(3)})\}$$

$$= 1.5253 + \frac{0.2}{2}[(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8851})]$$

$$= 1.8851$$

Hence at x = 0.6, y_3 = 1.8851

□ □ □

Numerical Solution for Differential Equations [Runge – Kutta Method]

Q.1. Using Runge – Kutta method find an approximate value of y for x = 0.2 in step of 0.1 if $\frac{dy}{dx} = x + y^2$ given y = 1 when x = 0

Ans.: Here $f(x, y) = x + y^2$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

$$K_1 = hf(x_0, y_0) = 0.1[0 + 1]$$

$$= 0.1 \text{ ……….} \qquad \text{\_ \_ \_ (1)}$$

$$K_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1)$$

$$= 0.1\left[\left(0 + \frac{1}{2}(0.1)\right) + \left(1 + \frac{1}{2} \times 0.1152\right)^2\right]$$

$$= 0.1152 \qquad \text{\_ \_ \_ (2)}$$

$$K_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2)$$

$$= 0.1\left[\left(0 + \frac{1}{2}(0.1)\right) + \left\{1 + \left(\frac{1}{2} \times 0.1152\right)\right\}^2\right]$$

$$= 0.1168 \qquad \text{\_ \_ \_ (3)}$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= 0.1\left[\ 0 + 0.1\ +\ 1 + 0.1168\ ^2\right]$$

$$= 0.1347 \qquad \text{\_ \_ \_ (4)}$$

and

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}\ 0.1 + 2(0.1152) + 2(0.1168) + 0.1347$$ {using equation (1), (2), (3) and (4)}

$= 0.1165$

Hence $y_1 = y_0 + K = 1 + 0.1165$

$= 1.1165$ ___ (5)

Again $x_1 = x_0 + h = 0.1$, $y_1 = 1.1165$, $h = 0.1$

Now

$$K_1 = hf(x_1, y_1)$$

$$= 0.1\left[0.1 + (1.1165)^2\right]$$

$$= 0.1347 \qquad \text{\_\_\_ (6)}$$

$$K_2 = hf\left[x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_1\right]$$

$$= 0.1\left[\left\{0.1 + \frac{1}{2}(0.1)\right\} + \left\{1.1165 + \frac{1}{2}(0.1347)\right\}^2\right]$$

$$= 0.1551 \qquad \text{\_\_\_ (7)}$$

$$K_3 = hf\left[x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_2\right]$$

$$= 0.1\left[\left\{0.1 + \frac{1}{2}(0.1)\right\} + \left\{1.1165 + \frac{1}{2}(0.1551)\right\}^2\right]$$

$$= 0.1576 \qquad \text{\_\_\_ (8)}$$

$$K_4 = hf\; x_1 + h, y_1 + K_3$$

$$= (0.1)\left[\; 0.1 + 0.1 \; + \; 1.1165 + 0.1576 \;^2\right]$$

$$= 0.1823 \qquad \text{\_\_\_ (9)}$$

and

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}\; 0.1347 + 2(0.1551) + 2(0.1576) + 0.1823$$ {using equation (6), (7), (8) and (9)}

$$= 0.1570$$

Hence

$$y(0.2) = y_2 = y_1 + K$$

$= 1.1165 + 0.1570$

$= 1.2735$

which is required solution.

Q.2. Use Runge-Kutta method to solve y' = x y for x = 1.4. Initially x = 1, y = 2 (tale h = 0.2).

[BCA Part II, 2007]

Ans.: (i) Here $f(x, y) = xy$, $x_0 = 1$, $y_0 = 2$, $h = 0.2$

$$K_1 = hf(x_0, y_0)$$

$$= 0.2[\,1 \times 2]$$

$$= 0.4$$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$$

$$= 0.2\left[\left(1+\frac{0.2}{2}\right)\times\left(2+\frac{0.4}{2}\right)\right]$$

$$= 0.2\left[\;1+0.1\;\times\;2+0.2\;\right]$$

$$= 0.2\left[\;1.1\quad 2.2\;\right]$$

$$= 0.484$$

$$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$$

$$= 0.2\left[\left(1+\frac{0.2}{2}\right)x\left(2+\frac{0.484}{2}\right)\right]$$

$$= 0.49324$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$= 0.2\left[\;1+0.2\;\times\;2+0.49324\;\right]$$

$$= 0.5983776$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}\;\;0.4+2(0.484)+2(0.49324)+0.5983776$$

$$= 0.4921429$$

$$y_1 = y_0 + K$$

$= 2 + 0.4921429$

$= 2.4921429$

(ii) $x_1 = x_0 + h = 1 + 0.2 = 1.2$, $y_1 = 2.4921429$ and $h = 0.2$

$K_1 = hf(x_1, y_1)$

$= 0.2[(1.2)(2.4921429)]$

$= 0.5981143$

$$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$$

$$= 0.2\left[\left(1.2 + \frac{0.2}{2}\right) \times \left(2.4921 + \frac{0.5981143}{2}\right)\right]$$

$= 0.81824$

$$K_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2})$$

$$= 0.2\left[\left(1.2 + \frac{0.2}{2}\right) \times \left(2.4921 + \frac{0.81824}{2}\right)\right]$$

$= 0.7543283$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$$= 0.2\left[\ 1.2 + 0.2\ \times\ 2.4921 + 0.7543\ \right]$$

$= 0.9090119$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$= 0.7753$

$y_2 = y_1 + K$

$= 2.4921 + 0.7753$

$= 3.26752$

$\therefore\ y(1.4) = 3.26752$

Multiple Choice Questions

1) Gauss Forward interpolation formula involves
(a) Even differences above the central line and odd differences on the central line.
(b) Even differences below the central line and odd differences on the central line.
(c) Odd differences below the central line and even differences on the central line.
(d) Odd differences above the central line and even differences on the central line.

The answer is (c)

2) The nth divided difference of a polynomial degree n is
(a) Zero (b) a Constant (c) a Variable (d) none of these

The answer is (b)

3) Gauss forward interpolation formula is used to interpolate values of y for
(a) $0 < p < 1$ (b) $-1<1<0$ (c) $0<p<$-infinite (d) -infinite$<p<0$

The answer is (a)

4) Newton's backward interpolation formula is used to interpolate the values of y near the
(a) Beginning of a set of tabulated values
(b) End of a set of tabulated values
(c) Center of a set of tabulated values
(d) None of the above

The answer is (b)

5) Langrange's interpolation formula is used for
(a) Unequal intervals (b) Equal intervals (c) Double intervals (d) None of these

The answer is (a)

6) Bessel's formula is most appropriate when p lies between
(a) -0.25and 0.25 (b) 0.25 and 0.75 (c) 0.25 and 1.00 (d) 0.75 and 1.00

The answer is (b)

7) Form the second divided difference table for the following data:

X:	5	7	11
Y:	150	392	1452

(a) 121
(b) 36
(c) 24
(d) 72

The answer is (c)

8) Extrapolation is defined as
(a) Estimate the value of a function within any two values given
(b) Estimate the value of a function inside the given range of values

(c) None of the above
(d) Estimate the value of a function outside the given range of values Both of the above

The answer is (d)

9) The second divided difference of f(x) = 1/x, with arguments a,b,c is
(a) abc (b) 1/abc (c) a/bc (d) b/ac

The answer is (b)

10) The following method is used to estimate the value of x for a given value of y(which is not in the table).
(a) Forward interpolation (b) Backward interpolation (c) iterative method (d) none of these

The answer is (c)

11) The bisection method for finding the roots of an equation f(x) = 0 is
(a) $X_{(n+1)} = ½(X_n + X_{n-1})$
(b) $X_{(n-1)} = ½(X_{n-1} + X_n)$
(c) $X_{(n+1)} = ½(X_n + X_n)$
(d) $X_{(n-1)} = ½(X_n + X_n)$

The answer is (a)

12) The bisection method of finding roots of nonlinear equations falls under the category of a (an) _________ method.
(a) open
(b) bracketing
(c) random
(d) graphical

The answer is (b)

13) If for a real continuous function f(x), f(a) f(b) < 0, then in the range of [a,b] for f(x) =0, there is (are)
(a) one root
(b) an undeterminable number of roots
(c) no root
(d) at least one root

The answer is (d)

14) For an equation like $X^2 = 0$, a root exists at X = 0. The bisection method cannot be adopted to solve this equation in spite of the root existing at X = 0 because the function $f(x) = X^2$
(a) is a polynomial
(b) has repeated roots at X = 0
(c) is always non-negative
(d) has a slope equal to zero at X = 0

The answer is (c)

15 Which of the following methods require(s) to have identified a bracket [*a b*] where *f*

changes of sign (f(a) f(b) < 0)?
(a) Newton–Raphson method
(b) Bisection method
(c) False Position method
(d) Secant method

The answer is (b)

16) In case of bisection method, the convergence is
(a) linear (b) quadratic (c) very slow (d) none of the above

The answer is (c)

17) In Regula-falsi method, the first approximation is given by
(a) X2 = (X0f(x1) –X1f(x0))/ f(x1) – f(x0)
(b) X2 = (X0f(x0) –X1f(x1))/ f(x1) – f(x0)
(c) X2 = (X0f(x1)+X1f(x0))/ f(x1) – f(x0)
(d) X2 = (X0f(x0) +X1f(x1))/ f(x1) – f(x0)

Solution
The answer is (a)

18) While finding the root of an equation by the method of false position, the number of iterations can be reduced
(a) start with larger interval
(b) start with smaller interval
(c) start randomly
(d) None of the above

The answer is (b)

19) The interval in which a real root of the equation X^3-2X-5 = 0 lies is
(a) (3,4) (b) (1,2) (c) (0,1) (d) (2,3)

The answer is (d)

20) The rate of convergence of the false position method to the bisection method is
(a) slower (b) faster (c) equal (d) none of the above

The correct answer is (b)

21) The order of convergence in Newton-Raphson method is
(a) 2 (b) 3 (c) 0 (d) none

The answer is (a)

22) The Newton raphson algorithm for finding the cube root of N is
(a) $X_{n+1} = 1/3(2X_n + N/x_n^2)$
(b) $X_{n+1} = 1/2(2X_n + N/x_n^2)$
(c) $X_{n+1} = 1/3(3X_n + N/x_n^2)$
(d) $X_{n+1} = 1/3(X_n + Nx_n^2)$

The answer is (a)

23) If $f(x) = 0$ is an algebraic equation, the Newton Raphson method is given by $X_{n+1} = X_n - f(x_n)/?$
(a) $f(x_{n-1})$ (b) $f'(x_{n-1})$ (c) $f'(x_n)$ (d) $f''(x_n)$

The answer is (c)

24) Newton's iterative formula to find the value of sqrt(N) is
(a) $X_{n+1} = \frac{1}{2}(X_n+N/X_n)$
(b) $X_{n+1} = \frac{1}{2}(2X_n+N/X_n)$
(c) $X_{n+1} = \frac{1}{2}(X_n+NX_n)$
(d) $X_{n+1} = \frac{1}{2}(X_n+N/2X_n)$

The answer is (a)

25) Newton-Raphson formula converges when
(a) Initial approximation is chosen sufficiently close to the root
(b) Initial approximation is chosen far from the root
(c) Initial approximation starts with zero
(d) Initial approximation is chosen randomly

The answer is (a)

26) In case of Newton Raphson method the convergence is
(a) linear (b) quadratic (c) very slow (d) none of the above

The answer is (b)

27) The Newton Raphson method fails when
(a) $f'(x)$ is negative (b) $f'(x)$ is too large (c) $f'(x)$ is zero (d) Never fails

The answer is (c)

28) The rate of convergence of Newton-Raphson method to false position method is
(a) slower (b) faster (c) equal (d) none of the above

The answer is (b)

29) Newton's method is useful when the graph of the function while crossing the x-axis is nearly
(a) horizontal (b) inclined (c) vertical (d) none of the above

The answer is (c)

30) Using Newton's method the root of $X^3 = 5X - 3$ between 0 and 1 correct to two decimal places is
(a) 0.677 (b) 0.557 (c) 0.765 (d) 0.657

The answer is (d)

31) The Secant method once converges, its rate of convergence is
(a) 1 (b) 1.5 (c) 1.9 (d) 1.6

The answer is (d)

32) In case of Successive Approximation method the convergence is

(a) linear (b) quadratic (c) very slow (d) none of the above

The answer is (a)

33) If the roots of the given polynomial are real and distinct then which method is quite useful
(a) bisection method (b) false position method
(c) newton-raphson (d) graeffe's root squaring

The answer is (d)

34) If all the roots of the given equation are required then the method used is
(a) false position (b) bisection method
(c) lin-baristow (d) secant

The answer is (c)

35) The equation $f(x) = 2X^7 – X^5 + 4X^3 – 5 = 0$ cannot have negative roots more than
(a) 1 (b) 0 (c) 2 (d) 4

The answer is (c)

36) If X is the true value of a quantity and X1 is its approximate value, then the relative error is
(a) |X1-X| / X1 (b) |X-X1| / X (c) |X1 / X| (d) X / |X1-X|

The answer is (a)

37) Errors present in the statement of a problem before its solution are called
(a) truncation error (b) rounding errors
(c) inherent errors (d) relative error

The answer is (c)

38) Errors caused by using approximate results or on replacing an infinite process by a finite one is
(a) rounding error (b) truncation errors
(c) inherent errors (d) relative error

The answer is (b)

39) If a number is correct to n significant digits, then the relative error is less than or equal to
(a) ½ 10^{-n}
(b) 10^{-n}
(c) 2 10^{-n}
(d) $n10^{-n}$

The answer is (a)

40) Errors caused from the process of rounding off the numbers during the computation are called
(a) rounding errors (b) inherent errors (c) relative error (d) truncation error

The answer is (a)

41) If X is the true value of a quantity and X ' is its approximate value, then the absolute error E_a is given by

(a) |X-X '| (b) |X+X '| (c) |X '-X| (d) |X '+X|

The answer is (a)

42) The relative error E_r is given by, where X is the true value

(a) $E_a + X$ (b) $E_a – X$ (c) X/E_a (d) E_a/X

The answer is (d)

43) The mantissa in the normalized floating point number is made

(a) less than 1 and greater than or equal to -.1
(b) less than 1 and greater than or equal to .1
(c) less than -1 and greater than or equal to 1
(d) less than 1 and greater than or equal to -1

The answer is (b)

44) Add .6434E99 and .4845E99

(a) 1.1279E99
(b) 0.11279E99
(c) 0.1128E99
(d) overflow condition

The answer is (d)

45) Subtract .5424E3 from .5452E3

(a) .2800E1
(b) .0028E3
(c) .0280E2
(d) none of the above

The answer is (a)

46) Multiply .5543E12* .4111E-15

(a) .2278E12
(b) .2278E-15
(c) .2278E-3
(d) .2278E3

The answer is (c)

47) Divide .1000E5 from .9999E3

(a) .1000E2
(b) .1000E-2
(c) .1000E3
(d) .1000E5

The answer is (a)

48) A real root of the equation $2X – \log_{10}X = 7$ correct to three decimal places using successive approx. method is

(a) 2.789
(b) 3.789

(c) 2.987
(d) 3.987

The answer is (b)

49) Determine f(3) where, $f(x) = 2X^5 – 13X^3 + 5X^2 – 11X – 6$ by synthetic division
(a) 140
(b) 153
(c) 49
(d) 141

The answer is (d)

50) What is the correct transformation of the equation to the form X = g(X) ? where $f(x) = X^3 + X^2 -1$
(a) X / sqrt(X+1)
(b) sqrt (X + 1) / X
(c) sqrt (X + 1)
(d) 1 / sqrt(X+1)

The answer is (d)

Bibliography

1. Computer Oriented Numerical Methods by R.S Salaria, fourth Edition, Khanna Book publishing Co.(p) LTD.
2. Numerical Analysis by Dr M.C. Goyal, Dr D.C.Sharma and Dr Savita Jain, RBD Publications