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Concept based notes

Basic Maths

BCA

Mr Rahul

Deptt. of IT

Biyani Girls College, Jaipur

Biyani's Group of Girls' Colleges

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Biyani Group of Colleges

Sector-3, Vidhyadhar Nagar,

Jaipur-302 023 (Rajasthan)

Ph : 0141-2338371, 2338591-95 • Fax : 0141-2338007

E-mail : acad@biyanicolleges.org

Website :www.gurukpo.com; www.biyanicolleges.org

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the "Teach Yourself" style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this endeavour. They played an active role in coordinating the various stages of this endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Unit 1

Functions, Limits and Continuity

Very Short Questions

Q.1. If $f(x)=\sqrt{25-x^2}$, find the value of $f(3), f(-4)$.

Ans. $f(3)=\sqrt{25-3^2}=\sqrt{25-9}=\sqrt{16}=4$

$f(-4)=\sqrt{25-(-4)^2}=\sqrt{25-16}=\sqrt{9}=3$

Q.2. If $f(x)=e^x$, prove that $f(x+y)=f(x).f(y)$

Ans. $f(x)=e^x$

$f(x+y)=e^{x+y} \Rightarrow e^x.e^y \Rightarrow f(x).f(y)$

Q.3. Evaluate $x \xrightarrow{\lim} 1 \quad x^2+3x+3$

$=1^2+3.1+3$

$=1+3+3=7$

Q.4. Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$

Ans. The given function is $f(x)=2x^2-1$

at $x=3$, $f(x)=f(3)=2\times3^2-1=17$

$\lim_{x\to3} f(x)=\lim_{x\to3}(2x^2-1)=2\times3^2-1=17$

$\therefore \lim_{x\to3} f(x)=f(3)$

Thus f is continuous at x = 3

Q.5. Find the domain of

$f(x)=\frac{x}{}$

Ans. $f(x)$ will be undefined if $3x - 4 = 0$ or $x = \frac{4}{3}$ so domain of $f = R\left\{\frac{4}{3}\right\}$

Q.6. The equation $y = (x)$, with $y \geq = 0$, represent y as function of x.

Ans. Yes

Q.7. The equation $x = |y|$, with $x >= 0$, represent y as a function of x.

Ans. False, solve for y to find that $y = |x|$ or $y = -|x|$ for one value of the independent variable x we have two values of the dependent variable y.

Q.8. R be the set of real number. If $f : R \to R, f(x) = x^2$ & $g : R \to R$, $g(x) = 2x + 1$ then find fog & gof show that $fog \neq gof$.

Ans. We know that $gof(x) = g[f(x)]$

$\Rightarrow$ $g(x^2) = 2(x^2) + 1 = 2x^2 + 1$ [since $g(x) = 2x + 1$]

and $fog = f[g(x)]$

$\Rightarrow$ $f(2x+1) = (2x+1)^2$ [since $f(x) = x^2$]

Q.9. Let $A = \{1,2,3,4\}, B = \{a, b, c, d\}, C = \{x, y, z\}$ consider the function $f : A \to B$ and $g : B \to C$ defined by

$f = \{(1,a),(2,c),(3,b),(4,a)\}$

$g = \{(a,x),(b,x),(c,y),(d,y)\}$.

Find gof

Sol. $(gof)(1) = g(f(1)) = g(a) = x$

$(gof)(2) = g(f(2)) = g(c) = y$

$(gof)(3) = g(f(3)) = g(b) = x$

$(gof)(4) = g(f(4)) = g(a) = x$

so $gof = \{(1,x),(2,y),(3,x),(4,x)\}$

Q.10. Find: $\lim_{x \to 1} 4x^2 + 3x + 2$.

Sol. Here

i.e. LHL at 'a' $= f(a) = RHL$ at 'a'

Q.15. Evaluate $x \xrightarrow{\lim} o \dfrac{x}{\sqrt{1+x}} - 1$

Solution : (Rationalizing)

$$\lim_{x\to 0} \frac{x}{\sqrt{1+x}-1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$\lim_{x\to 0} \frac{x\left(\sqrt{1+x}+1\right)}{1+x-1} = \lim_{x\to 0} \left(\sqrt{1+x}+1\right)$$

$= 1 + 1 = 2$

Q.16. Define the following function.

(i) Identify function

(ii) Constant function

(iii) Logarithm function

(iv) Hyperbolic function

Ans. (i) $f(x) = x$ (ii) $f(x) = c$

(iii) $f(x) = \log x$ (iv) $f(x) = \sin hx$

Q.17. Evaluate

$$\lim_{x\to\infty} \left(\frac{1+2+3+........n}{n^2} \right)$$

$$= \lim_{x\to\infty} \frac{n(n+1)}{2n^2}$$

$$= \lim_{x\to\infty} \left(\frac{(n+1)}{2n} \right) = \lim_{x\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{2}$$

Q.18. Evaluate the given limit $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}, \quad a, b \neq o$

Ans. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}, \quad a, b \neq o$

divide & multiply numerator by ax

divide & multiply denominator by bx.

$$\lim_{x\to 0} \frac{= \frac{\sin ax}{ax}.ax}{\frac{\sin bx}{bx}.bx}$$

$$= \frac{\frac{a}{b} a \lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)}{b \lim_{x\to 0}\left(\frac{\sin bx}{bx}\right)}$$

$\left[\begin{array}{l} x \to o \Rightarrow ax \to o \\ x \to o \Rightarrow bx \to o \end{array}\right.$

$$= \frac{a}{b} \times \frac{1}{1}$$

$\left[\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right.$

$$= \frac{a}{b}$$

Q.19. Evaluate the given limit

$$\lim_{x\to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, \quad a+b+c \neq 0$$

Ans. $\frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a} = \frac{a+b+c}{a+b+c} = 1$

Q.20. Evaluate given limit

$$\lim_{x\to 0} \frac{ax+b}{cx+1} = \frac{a(o)+b}{c(o)+1}$$

$$= b$$

Q.21. If $f(2) = 4$ and $f'(2) = 1$, then find.

$$\lim_{x\to 2} \frac{xf(2) - 2f(x)}{x-2}$$

Solution: $\lim_{x\to 2} \frac{xf(2) - 2f(x)}{x-2}$

$$\lim_{x\to 2} \frac{xf(2) - 2f(x) + 2f(2) - 2f(2)}{x-2}$$

[add and subtract 2f(2) in numerator

$$= \lim_{x\to 2} \frac{xf(2)-2f(2)+2f(2)-2f(x)}{(x-2)}$$

[Rearrange 2f(2)

$$= \lim_{x\to 2} \frac{[f(2)(x-2)-2(f(2)-f(x)]}{x-2}$$

[take f(2) common take 2 common

$$= \lim_{x\to 2} \frac{(x-2)f(2)}{x-2} - \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$$

$$= f(2)-2f'(2) \qquad \left[\because f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}\right.$$

$$= 4-2\times 1 \qquad [\because f(2) = 4 \text{ and } f'(2)=1 \text{ from ques.}]$$

$=2$

Q.22. Explain the continuity of the function.

$$f(x)\begin{cases} \dfrac{|\sin x|}{x}, & x \neq o \\ 1, & x = o \end{cases}$$

$$f(o)=1$$

$$\lim_{n\to 0} f(o+n) = \lim_{n\to 0} \frac{|\sin(o+n)|}{o+n} - \lim_{n\to 0} \frac{\sin n}{n} = 1$$

$$\lim_{n\to 0} f(o-n) = \lim_{n\to 0} \frac{|\sin(o-n)|}{o-n} - \lim_{n\to 0} \frac{\sin n}{-n} = -1$$

$$f(o) = f(o+n) \neq f(o-n)$$

$$1 = 1 \neq -1$$

$f(x)$ is not continuous.

Q.23. Show that $f(x) = x^2$ is continuous

at x = 3

$$f(x) = x^2$$

$f(3)=3^2=9$

$\lim_{n\to 0} f(3+n)=\lim_{n\to 0}(3+n)^2=\lim_{n\to 0}\left(9+n^2+6n\right)=9$

$\lim_{n\to 0} f(3-n)=\lim_{n\to 0}\ (3-n)^2=\lim_{n\to 0}\ \left(9+n^2-6n\right)=9$

$\therefore f(x)=f(x+n)=f(x-n)=9$

$\therefore f(x)$ is continuous at x = 3.

Q.24. Show that $\lim_{x\to 0}\frac{x}{|x|}$ does not exist

Sol. $f(o)=o$

$\lim_{n\to 0} f(o+n)=\frac{o+n}{|o+n|}=\lim_{n\to 0}\ \frac{h}{h}=\lim_{n\to 0}\ 1=1$

$\lim_{n\to 0} f(o-n)=\frac{o-n}{|o-n|}=\lim_{n\to 0}\ \frac{-n}{n}=-1$

$\because f(o)\neq f(o+n)\neq f(o-n)$

$f(x)$ is not continuous.

Short Questions

Q.1 Find the range of $f(x)=\frac{2+x}{2-x}$

Sol. $f(x)=y$

$\frac{2+x}{2-x}=y$

$2+x=y(2-x)\Rightarrow 2+x=2y-xy\Rightarrow x+xy=2y-2$

$x(1+y)=2(y-1)\Rightarrow x=\frac{2(y-1)}{y+1}$

x will be undefined if $y+1=0,\ y=-1$

$\therefore$ Range of $f=R-\{-1\}$

Q. 2 : Let A = $\{-2,-1,0,1,2\}$ and B = {0, 1, 2, 3, 4, 5} A function is defined from set A to set B. If $f(x) = x^2$. find the domain, co-domain and range of the function.

Sol. : $f(-2) = (-2)^2 = 4$

$f(-1) = (-1)^2 = 1$

$f(0) = (0)^2 = 0$

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

Here, domain = A = $\{-2, -1, 0, 1, 2\}$

co-domain = $\{0, 1, 2, 3, 4, 5\}$

range = $\{0, 1, 4\}$

Q. 3 : A function F : R $\rightarrow$ R defined as $f(x) = 5x - 3$ find its inverse.

Sol. : Let's considers

$$y = f(x) \Rightarrow x = f^{-1}(x)$$

$$\Rightarrow \quad y = 5x - 3$$

$$\Rightarrow \quad 5x = y + 3$$

$$\Rightarrow \quad x = y + 3/5 \Rightarrow f'(y) = \frac{y+3}{5}$$

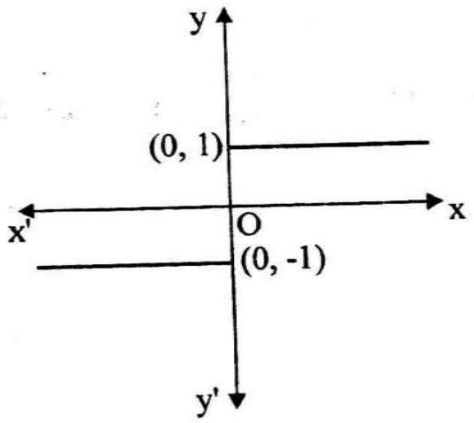
Q. 4. What is sigmum function ?

Sol. Signum function :- The function is defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$or\ f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

Domain = R and Range = {-1, 0, 1}.



Q. 5 : What is reciproal function ?

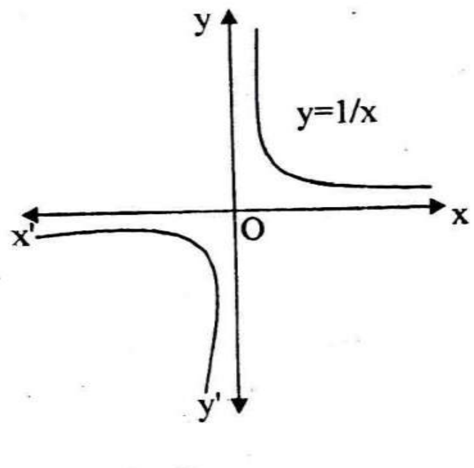
Sol.. : Reciprocal function :-

The function defined by $f(x)=\frac{1}{x}$ is called the reciprocal function.

The function is not defined for $x = 0$.

Domain $= R - \{0\}$ and Range

Also, .



Q. 6 : What is modulus function ?

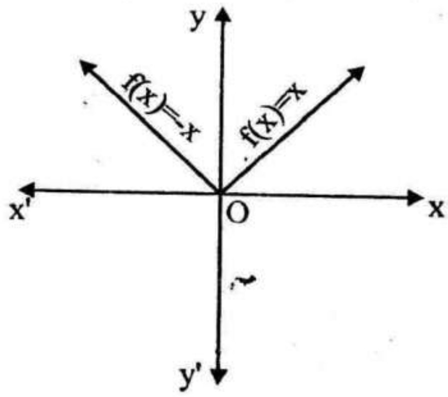
Sol. : Modulus function :-

A function is called the modulus function if defined by

$$f(x)=|x|=\begin{cases} x, & if \ x \geq 0 \\ -x, & \text{When } x<0 \end{cases}$$

We can say that modulus of every number is a non-negative real number, so

Domain $=$ *R and Range* $=$ non-negative real numbers.



Q. 7 : *Evalute* $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

Sol.: Here

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$$

$$= \lim_{x \to 1} (x^2 + x + 1)$$

$$= (1^2 + 1 + 1)$$

$$= 3$$

Q. 8 : Evalute $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 + x - 6}$

Sol: Here $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 + x - 6}$ $\quad \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 2} \frac{(x-2)(x+4)}{(x-2)(x+3)}$$

$$= \lim_{x \to 2} \frac{(x+4)}{(x+3)}$$

$$= \frac{2+4}{2+3} = \frac{6}{5}$$

9 : $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$

.: $\lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{x}}{\frac{x+1}{x}}$ **(divide both numerator and denominator by x)**

$\lim_{x\to\infty} \frac{\frac{\sqrt{x^2+1}}{x^2}}{\frac{x+1}{x}}$ **(x equals to $\sqrt{x^2}$)**

$$\lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}{\frac{x}{x}+\frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{1+\sqrt{\frac{1}{x^2}}}{1+\frac{1}{x}}$$

$$\frac{\sqrt{1+\frac{1}{\infty^2}}}{1+\frac{1}{\infty}} \Rightarrow \frac{\sqrt{1+0}}{1+0} = 1 \left[\because \frac{1}{\infty} = 0\right]$$

Q. 9 : Find all the points of discontinuity of f, where f is defined by

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

Sol. : The given function is $f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

$\because$ We know that $x < 0 \Rightarrow |x| = -x$

$\therefore$ The given function can be written as

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$\Rightarrow$ $f(x) = -1$ for all $x \in R$

let C be any real number. Then $\lim_{x \to c} f(x) = \lim_{x \to c} f(x) = -1$

Also $f(c) = -1 = \lim_{x \to c} f(x)$

$\therefore$ function is continuous.

Q. 10. Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$

Sol. : $\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + - 1\right]}{x}$ [used expansion of e^x]

$$= \lim_{x \to 0} \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...}{x}$$ [cut 1 and -1]

take x common from numerator

$$= \lim_{x \to 0} \frac{x\left[1 + \frac{x}{2!} + \frac{x^2}{3!} + ...\right]}{x}$$

$$= \lim_{x \to 0} 1 + \frac{x}{2!} + \frac{x^2}{3!} + ...$$ [cut x in numerator & denominator]

Put the limit

$= 1$

Q.11 Determine the value of k for following function is continues at $x = 1$

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ K - 2x, & 1 \leq x \leq 2 \end{cases}$$

Sol. : $f(1-0) = f(1+0) = f(1)$

$\because$ f is continuous at $x = 1$

$f(1) = K - 2$

LHL at $x = 1$

$$\lim_{n \to 0} f(1-n) = \lim_{n \to 0} (1-n) = 1$$

$f(1-0) = 1$

For continuity $K - 2 = 1 \Rightarrow K = 3$

12. Show that $f(x) = x^2$ is diffrentiable at $x = 1$ and find $f'(1)$

Sol. : We have,

$$(\text{L.H.D. at } x = 1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$(\text{LHD}) \text{ at } x = 1 = \lim_{h \to 0} \frac{f(1-n) - f(1)}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{f(1-n)^2 - 1^2}{-h} = \lim_{h \to 0} \frac{-2n + n^2}{-h}$$

$$= \lim_{n \to 0} 2 - n = 2$$

and (RHD at $x = 1$)

$$= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{h \to 0} \frac{f(1+n) - f(1)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{f(1+n)^2 - 1}{h} = \lim_{h \to 0} \frac{2n + n^2}{h}$$

$$= \lim_{h \to 0} 2 + h = 2$$

(L.H.D. at $x = 1$) = (RHD at $x = 1$) = 2

So, $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$

Q. 13 Check the continuity of the following function

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + c^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \textbf{at } x = 0$$

$$f(x) = 0, f(x+n) = \left(\frac{e^{1/x+n}}{1 + e^{1/x+n}} \right)$$

$$\lim_{h\to 0} f(x+n) = \lim_{h\to 0} \frac{e^{1/x+n}}{e^{1/x+n}}\left\{\frac{1}{e^{1/x+n}+1}\right\}$$

$$= \lim_{h\to 0}\left[\frac{1}{e^{-1/n}+1}\right] = \frac{1}{0+1} = \boxed{1}$$

$$\lim_{n\to 0} f(x-n) = \lim_{h\to 0}\left[\frac{e^{1/x-n}}{1+e^{1/x-n}}\right] = \lim_{n\to 0}\left[\frac{e^{-1/n}}{1+e^{-1/n}}\right]$$

$$= \frac{0}{1+0} = \boxed{1}$$

$f(x) = 0$

$f(x+n) = 1 \quad \Rightarrow f(x) \neq f(x-n) = f(x-n)$

$f(x-n) = 1$

$\Rightarrow$ function is discontinuous

Q.14 Prove that

$$f(x) = \begin{cases} x-1, & \text{if } 1 \le x \le 2 \\ 2x-3, & \text{if } 2 \le x \le 3 \end{cases}$$ **Parth publication exrcise Ques.**

is continuous at $x = 2$

Sol.

$f(x) = x-1 \qquad f(x+n) = 2x-3$

$f(2) = 2-1 \qquad \lim_{n\to 0} f(2+n) = \lim_{n\to 0} 2(2+n)-3$

$f(2) = 1 \qquad = \lim_{n\to 0} 4+2n-3$

$$= \lim_{n\to 0} 4+2n-3$$

$$f(2+0) = \lim_{n\to 0} 1+2n = 1$$

$f(x-n) = x-1$

$\lim_{n\to 0} f(2-n) = \lim_{n\to 0}(2-n)-1$

$\lim_{n\to 0} f(2-n) = \lim_{n\to 0} 2-n-1$

$$= \lim_{n\to 0} 1 - h = 1$$

$$f(x) = f(x+n) = f(x-n)$$

$\therefore$ function is continuous

Q. 15 Evaluate $\lim_{x\to a} \frac{\sqrt{x}+\sqrt{a}}{x+a}$

$$= \frac{\sqrt{a}+\sqrt{a}}{a+a} = \frac{2\sqrt{a}}{2a}$$

$$= \frac{\sqrt{a}}{a}$$

$$= \frac{1}{\sqrt{a}}$$

Q. 16 Evaluate the left hand limit and right hand limit of function given below:

$$f(x) = \begin{cases} \frac{|x-5|}{x-5}, & if\ x \neq 5 \\ 0 \quad , if & x = 5 \end{cases} \text{ at } x = 5.$$

Solution: Left hand limit (LHL) at x = 5

$$= \lim_{x\to 5^-} f(x)$$

$$= \lim_{h\to 0} f(5-h)$$

$$= \lim_{h\to 0} \frac{|5-h-5|}{5-h-5} = \lim_{h\to 0} \frac{|-h|}{-h}$$

$$= \lim_{h\to 0} \frac{h}{-h} = \lim_{h\to 0} -1 = -1$$

Right hand limit (RHL) at x = 5

$$= \lim_{x\to 5^+} f(x)$$

$$= \lim_{h\to 0} f(5+h)$$

$$= \lim_{h\to 0} \frac{|5+h-5|}{5+h-5} = \lim_{h\to 0} \frac{|h|}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1.$$

Q. 17 If $f(x)=\begin{cases} 5x-4, & 0<x\le 1 \\ 4x^3-3x, & 1<x<2 \end{cases}$ show that $\lim_{x\to 1} f(x)$ exist.

So.: Left hand limit (LHL) at x = 1

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} 5(1-h)-4 = \lim_{h\to 0} 5-5h-4$$

$$= \lim_{h\to 0} 1-5h$$

$$= 1-5(0) \quad \text{(using limit)}$$

$$= 1$$

Now,

Right hand limit (RHL) at x = 1

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} 4(1+h)^3 - 3(1+h)$$

$$= 4 - 3 = 1$$

Here

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$$

(L.H.L = R.H.L)

So $\lim_{x\to 1}$ exits at $x=1$

Q. 18 Show that $\lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$ **does not exist.**

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

$$= \lim_{h\to 0} \left(\frac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1} \right) \quad \text{using} \left(a^{-x} = \frac{1}{a^x} \right)$$

$$\left[\begin{array}{c} h \to 0 \Rightarrow \frac{1}{h} \to \infty \\ So, e^{1/h} \to \infty \Rightarrow \frac{1}{e^{1/h}} \to 0 \end{array} \right]$$

Right hand limit (RHL) at $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

$$= \lim_{h\to 0} \left(\frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}} \right)$$

(Dividing numerator and denomenator by $e^{1/h}$)

$$= \frac{1-0}{1+0}$$
$$= 1$$

it is clear that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0} f(x)$

i.e. $LHL \neq RHL$

So, $\lim_{x\to 0} f(x)$ does not exist.

Q. 19 A $f(x)$ be a function defined by $f(x) = \begin{cases} 4x-5, & if \quad x \le 1 \\ 2x-\lambda, & if \quad x > 1 \end{cases}$ **Find**

$$f(x)=\begin{cases}4x-5, & if\ \ x\le 1\\ 2x-\lambda, & if\ \ x>1\end{cases}$$

Now,

Left hand limit (LHL) at x =1

$$\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)$$

$$=\lim_{h\to 0} 4(1-h)-5$$

$$=\lim_{h\to 0}(-1-4h)=-1$$

Right hand limit (RHL) at $x=1$

$$\lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h)$$

$$=\lim_{h\to 0} 2(1+h)-\lambda$$

$$=(2-\lambda)$$

Since $\lim_{x\to 1} f(x)$ exists, So

LHL = RHL

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)$$

$$\Rightarrow \quad 1=2-\lambda \quad \Rightarrow \quad \lambda=1$$

Q. 20 Evalute $\lim_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1}$

Sol:- Here $\lim_{x\to 1}\dfrac{x^2-\sqrt{x}}{\sqrt{x}-1} \quad \left(\dfrac{0}{0}\, form\right)$

$$=\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}\times\frac{x^2+\sqrt{x}}{x^2+\sqrt{x}}$$

$$=\lim_{x\to 1}\frac{x^4-x}{\left(\sqrt{x}-1\right)\left(x^2+\sqrt{x}\right)} \quad \left(\frac{0}{0}\, form\right)$$

$$= \lim_{x \to 1} \frac{x\left(x^3 - 1\right)}{\left(\sqrt{x} - 1\right)\left(x^2 + \sqrt{x}\right)} \quad \left(\frac{0}{0} form\right)$$

$$= \lim_{x \to 1} \frac{x(x-1)\left(x^2 + x + 1\right)}{\left(\sqrt{x} - 1\right)\left(x^2 + \sqrt{x}\right)} \qquad \left(\frac{0}{0} form\right)$$

$$\left[\because a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)\right]$$

$$= \lim_{x \to 1} \frac{x\left(\sqrt{x} + 1\right)\left(x^2 + x + 1\right)}{\left(x^2 + \sqrt{x}\right)} = \frac{1(1+1)(1+1+1)}{(1+1)} = \frac{2(3)}{2}$$

$$= 3$$

Q. 21 Test the continuity of the function f(x) at the origin:

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$$

Sol. Here $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

So, $f(0) = 1$. (Using the value of the function)

Now,
RHL at x = 0

$$\lim_{x \to 0+} f(x)$$
$$= \lim_{h \to 0} f(0 + h)$$
$$= \lim_{h \to 0} \frac{h}{|h|}$$
$$= \lim_{h \to 0} \frac{h}{h}$$
$$= 1.$$

LHL at x = 0

$$\lim_{x \to 0-} f(x)$$

$$= \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{-h}{|h|}$$

$$= \lim_{h \to 0} \frac{-h}{h}$$

$$= -1.$$

Now here $f(x) = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

and therefore, $\lim_{x \to 0} f(x)$ does not exist.

So, $f(x)$ is discontinuous at $x = 0$.

Q. 22 Show that the exponential function is continuous.

Sol: Let's consider $f(x) = e^x$

Now,

RHL at h = 0

$$\lim_{h \to 0^+} f(x)$$

$$= \lim_{h \to 0^+} f(x+h)$$

$$= \lim_{h \to 0} e^{(x+h)}$$

$$= \lim_{h \to 0} e^x . e^h$$

$$= e^x . e^0 \qquad (\because e^0 = 1)$$

$$= e^x . 1$$

$$= e^x$$

LHL at h = 0

$$\lim_{h \to 0^-} f(x)$$

$$= \lim_{h \to 0} f(x-h)$$

$$= \lim_{h\to 0} f(x-h) = \lim_{h\to 0} e^{x-h}$$

$$= \lim_{h\to 0} e^{x}.e^{-h} = \lim_{h\to 0} \frac{e^x}{e^h}$$

$$= \frac{e^x}{e^0} = \frac{e^x}{1} = e^x$$

Hence $f(x) = \lim_{h\to 0^+} f(x) = \lim_{h\to 0^-} f(x)$

So, Exponetial function is continous.

Long Questions

Q.1 Differentiate following by first principle method.

I. $f(x) = 1/x$

$$\frac{dy}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \frac{\frac{1}{x+\delta x} - \frac{1}{x}}{\delta x}$$

$$= \frac{\frac{x-(x+\delta x)}{x(x+\delta x)}}{\delta x}$$

$$= \frac{-\delta x}{x(x+\delta x)\delta x}$$

$$= \frac{-1}{(x+\delta x)x}$$

In the limit as $\delta x \to 0$ this becomes

$$\frac{dy}{dx} = -\frac{1}{x^2}.$$

So the derivative of $y = \frac{1}{x}$ with respect to x is $-\frac{1}{x^2}$

OR

Q.1. Evaluate $\lim_{x \to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$

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Differentiate using first principles

$$y = x^2 + 2x + 3$$

let $f(x) = x^2 + 2x + 3$

so $f(x + \delta x) = (x + \delta x)^2 + 2(x + \delta x) + 3$

expanding and simplifying gives

$$f(x + \delta x) = x^2 + 2x\delta x + \delta x^2 + 2x + 2\delta x + 3$$

We have everything we need to substitute into the formula for differentiation from first principles.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Substituting in the first principles formula gives:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{(\cancel{x^2} + 2x\delta x + \delta x^2 + \cancel{2x} + 2\delta x + \cancel{3}) - (\cancel{x^2} + \cancel{2x} + \cancel{3})}{\delta x} \right)$$

We notice that Everything in the 2nd bracket will cancel out with the first bracket.

This will Always be the case!!!

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{2x\delta x + \delta x^2 + 2\delta x}{\delta x} \right)$$

Notice that the top has a **Common factor** of δx.
So factorise the top and cancel δx

This will Always be the case!!!!

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{\cancel{\delta} x(2x + \delta x + 2)}{\cancel{\delta} x} \right)$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{2x + \delta x + 2}{1} \right)$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} (2x + \delta x + 2)$$

Now the **important final stage** is to find the limit as $\delta x \to 0$

$$\frac{dy}{dx} = 2x + 2$$

Q.2. Show that the function f(x) given by

$$f(x) = \begin{cases} \lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}, & if\ x \neq 0 \\ 0 & , if\ x = 0 \end{cases}$$

is discontinuous at x = 0.

Sol.: Given

$$f(x) = \begin{cases} \lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}, & if\ x \neq 0 \\ 0 & , if\ x = 0 \end{cases}$$

Now, f(x) = 0

Here, Now we find, Left hand limit (LHL) at $x = 0$

$$\lim_{x \to 0^-} f(x)$$

$$= \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

$$= \lim_{h \to 0} \left(\frac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1} \right) \quad using \left(a^{-x} = \frac{1}{a^x} \right)$$

$$\left[\begin{array}{c} h \to 0 \Rightarrow \frac{1}{h} \to \infty \\ So, e^{1/h} \to \infty \Rightarrow \frac{1}{e^{1/h}} \to 0 \end{array} \right]$$

= -1

Right hand limit (RHL) at $x = 0$

$$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

$$= \lim_{h \to 0} \left(\frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}} \right)$$

(Dividing numerator and denomenator by $e^{1/h}$)

$$= \frac{1-0}{1+0}$$
$$= 1$$

it is clear that $f(x) \neq \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0} f(x)$

So, $f(x)$ is not continuous at x = 0.

Q.3 Find $\frac{dy}{dx}$ from first principles if $y = 2x^2 + 3x$.

Find the slope of the tangent where $x = 1$ and also where $x = -6$.

Sol. Now $f(x) = x^2 + 4x$

So $\quad f(x+h) - (x+h)^2 + 4(x+h)$

$$-x^2 + 2xh + h^2 + 4x + 4h$$

Therefore

$$\frac{dy}{dx} - \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$- \lim_{h \to 0} \frac{\left[(x+h)^2 + 4(x+h) - [x^2 + 4x]\right]}{h}$$

$$- \lim_{h \to 0} \frac{\left[x^2 + 2xh + h^2 + 4x + 4h\right] - \left[x^2 + 4x\right]}{h}$$

$$- \lim_{h \to 0} \frac{2xh + h^2 + 4h}{h}$$

$$-\lim_{h\to 0}(2x+h+4) \quad -2x+4$$

OR

3. Prove that the following $f(x)=|x|+|x-1|$ is continuous at x = o, 1 but not differentiable.

Sol. $f(x)=|x|+|x-1|$

$$f(x)=\begin{cases}1-2x & ,x<o\\ 1 & ,o\le x<1\\ 2x-1 & ,x\ge 1\end{cases}$$

x = o $f(o)=1$

$f(o+n)=1$ and $f(o-n)=1-2(o-n)=1+2h$

RHL $=\lim_{n\to 0} f(o+n)=\lim_{n\to 0}(1)=1$

LHL $=\lim_{n\to 0} f(o-n)=\lim_{n\to 0}(1+2h)=1$

$$f(o)=RHL=LHL$$

Hence function is continuous at x = o

$$Rf'(o)=\lim_{n\to 0}\frac{f(o+n)-f(o)}{h}=\lim_{n\to 0}\left(\frac{1-1}{n}\right)=o$$

$$Lf'(o)=\lim_{n\to 0}\frac{f(o-n)-f(o)}{-h}$$

$$=\lim_{n\to 0}\frac{(1++2n-1)}{-h}$$

$$= -2$$

$Rf'(o)\ne Lf'(o)$

Hence function is not diffeentiable at x = 0

Again at $x=1$ $f(1)=2-1=1$

$f(1+h)=2(1+n)-1=1+2n$ and $f(1-n)=1$

RHL $=\lim_{n\to 0} f(1+n)=\lim_{n\to 0}(1+2n)=1$

LHL $= \lim_{n\to 0} f(1-n) = \lim_{n\to 0} f(1) = 1$

$$f(1) = RHL = LHL$$

Hence function is continuous at x = 1

$$Rf'(1) = \lim_{n\to 0} \frac{f(1+n)-f(1)}{h} = \lim_{n\to 0} \frac{1+2n-1}{h}$$

$$=2$$

$$Lf'(1) = \lim_{n\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{n\to 0} \frac{1-1}{-h} = o$$

$$Rf'(1) \neq Lf'(1)$$

Hence function is not differentiable at x = 1

Q.4. A function given by $f(x) = \begin{cases} x, \text{ if } x \geq 1 \\ x^2, \text{if } x < 1. \end{cases}$ **is a continuous function?**

Sol. Lets consider a be any real number. There are three possibilities.

Case I : If $a > 1$

$$f(a) = a.$$

Now RHL

$$\lim_{x\to a+} f(x)$$

$$= \lim_{x\to 0} f(a+h)$$

$$= \lim_{h\to 0} (a+h)$$

$$= a$$

LHL

$$\lim_{x\to a-} f(x)$$

$$= \lim_{x\to 0} f(a-h)$$

$$= \lim_{h\to 0} (a-h)$$

$$= a$$

So, $\lim_{x\to a+} f(x) = \lim_{a\to a-} f(x) = f(a).$

Hence, $f(x)$ is continuous at each $a > 1$.

Case II : If $a < 1$

Here , $f(a) = a^2$

RHL

$$\lim_{x \to a+} f(x)$$
$$= \lim_{h \to 0} f(a+h)$$
$$= \lim_{h \to 0} (a+h^2)$$
$$= a^2$$

LHL

$$\lim_{x \to a-} f(x)$$
$$= \lim_{x \to 0} f(a-h)$$
$$= \lim_{h \to 0} (a-h)^2$$
$$= a^2$$

$$So, \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a).$$

Hence, $f(x)$ is continuous at each $a < 1$.

Case III: If $a = 1$

$f(1) = 1$.

RHL

$$\lim_{x \to 1+} f(x)$$
$$= \lim_{h \to 0} f(1+h)$$
$$= \lim_{h \to 0} (1+h)$$
$$= 1$$

LHL

$$\lim_{x \to 1-} f(x)$$
$$= \lim_{h \to 0} f(1-h)$$
$$= \lim_{h \to 0} (1-h)^2$$
$$= 1$$

$$So, \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a).$$

Hence, $f(x)$ is continuous at each $a = 1$.

So, finally $f(x)$ is continuous at $x = a \ \forall a \in R$.

Hence, $f(x)$ is continuous.

OR

Q.4. Find the derivative of xe^x w.r.t. to x by first principle.

Sol. let $y = xe^x$

$$y + \delta y = (x + \delta x)e^{(x+\delta x)}$$

$$\delta y = (x + \delta x)e^{x+\delta x} - xe^x$$

$$\delta y = (x + \delta x)e^x.e^{\delta x} - xe^x$$

$$\delta y = e^x\left[(x + \delta x)e^{\delta x} - x\right]$$

$$\delta y = e^x\left[(x + \delta x)\left(1 + \delta x + \frac{(\delta x)^2}{2!} + ...\right) - x\right]$$

$$\delta y = e^x\left[x + \delta x + x\delta x + \delta x^2 + ... - x\right]$$

$$\frac{\delta y}{\delta x} = \frac{e^x}{\delta x}\left[\delta x + x\delta x + \delta x^2\right]$$

$$\frac{\delta y}{\delta x} = \frac{e^x}{\delta x}, \delta x[1 + x + \delta x]$$

$$\delta x \xrightarrow{\lim} o\frac{\delta y}{\delta x} = \delta x \xrightarrow{\lim} oe^x[1 + x + o]$$

$$\frac{dy}{dx} = e^x[1 + x]$$

Q.5. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

is a continuous function.

Sol. : The given function f is $f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x = 2$ and $x = 10$

Since f is continuous at $x = 2$, we obtain

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} f(x) = f(2)$$

$$\Rightarrow \quad \lim_{x\to 2}(5) = \lim_{x\to 2}(ax+b) = 5$$

$$\Rightarrow \quad 5 = 2a+b = 5$$

$$\Rightarrow \quad 2a+b = 5 \qquad \ldots(1)$$

Since f is continuous at $x = 10$, we obtain

$$\lim_{x\to 10} f(x) = \lim_{x\to 10} f(x) = f(10)$$

$$\Rightarrow \quad \lim_{x\to 10}(ax+b) = \lim_{x\to 10}(21) = 21$$

$$\Rightarrow \quad 10a+b = 21 = 21$$

$$\Rightarrow \quad 10a+b = 21 = 21$$

$$\Rightarrow \quad 10a+b = 21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

$$8a = 16$$

$$\Rightarrow \quad a = 2$$

By putting a = 2 in equation (1), we obtain

$$2\times 2+b = 5$$

$$\Rightarrow \quad 4+b = 5$$

$$\Rightarrow \quad b = 1$$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.

OR

Q.5. Differentiate the following w.r.t. x

(i) $\cos^3 x$

Ans. Let's consider

$$y = \cos^3 x$$

$$= (\cos x)^3.$$

Putting $\cos x = t$

So, $y = t^3$ and $t = \cos x$

$$\Rightarrow \frac{dy}{dt} = 3t^2 \text{ and } \frac{dt}{dx} = -\sin x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right).$$

$= (-3t^2 \sin x)$

Now,Putting the value of t

$= (-3 \sin x \cos^2 x).$

(ii) $e^{4\log y}$

Ans. Let's consider

$$y = e^{4\log x}$$

putting $4\log x = t$

$$y = e^t \text{ and } t = 4\log x$$

$$\Rightarrow \quad \frac{dy}{dt} = e^t \text{ and } \frac{dy}{dx} = \frac{4}{x}$$

since $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = e^t \times \frac{4}{x}$

putting the value of t

$$\frac{dy}{dx} = \frac{4e^{4\log x}}{x}$$

(iii) $\tan\sqrt{x}$

Ans. Let's consider $y = \tan\sqrt{x}$

Putting $\sqrt{x} = t$

So, $y = \tan t$ and $t = \sqrt{x}$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right).$$

$$= \left(\sec^2 t . \frac{1}{2\sqrt{x}}\right)$$

Now, Putting the value of t

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

(iv) $y = (x+3)^5$

Ans. Let's consider $y = (x+3)^5$

Putting $(x+3) = t$

$y = t^5$ and $t = x + 3$

$\Rightarrow \frac{dy}{dt} = 5t^4$ and $\frac{dt}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \left(\frac{dy}{dt} \times \frac{dt}{dx}\right) = 5t^4$

Now Putting the value of t

$= 5(x+3)^4$.

Q. 6. Differentiate x^3 by first principle method.

Ans. We know $f(x) = x^3$, and can calculate $f(x + \Delta x)$:

Start with : $f(x+\Delta x) = (x+\Delta x)^3$

Expand : $(x+\Delta x)^3$: $f(x+\Delta x) = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$

The slope formula : $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

Put in $f(x+\Delta x)$ and $f(x)$: $\frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

Simplify (x^3 and $-x^3$ cancel): $= -\frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$

Simplify more (divide through by Δx) : $= 3x^2 + 3x\,\Delta x + (\Delta x)^2$

And then as Δx heads towards 0 we get : $\frac{d}{dx}x^3 = 3x^2$

OR

Q.6. Draw the graph of the following (any 4)

1. Constant function

2. Identity

3. Modulus

4. Reciprocal

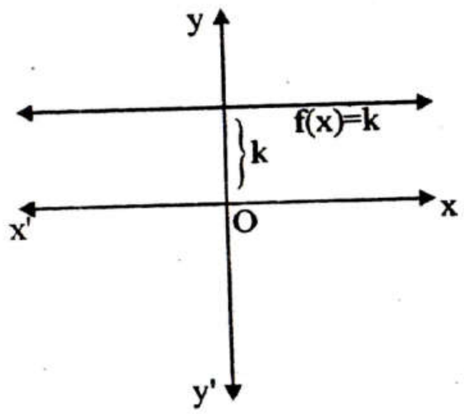
5. Signum

6. Square root

Ans. 1. Constant Function :-

Let c be a constant real number. Then, the function defined by $f(x) = c$ for all $X \in R$ is called constant function c.

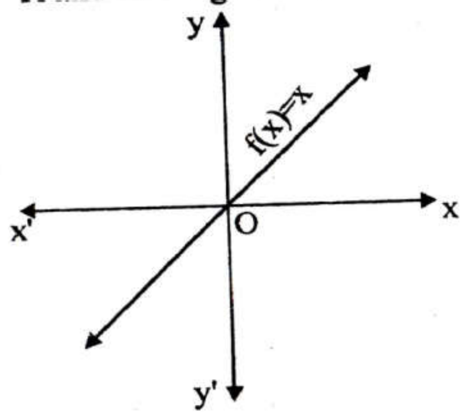
domain (f) = R and range (f) = $\{c\}$.



2. Identity function :

The function defined $f(x) = x$ for all $X \in R$ is called the identity function.

The domain = R and its range = R.



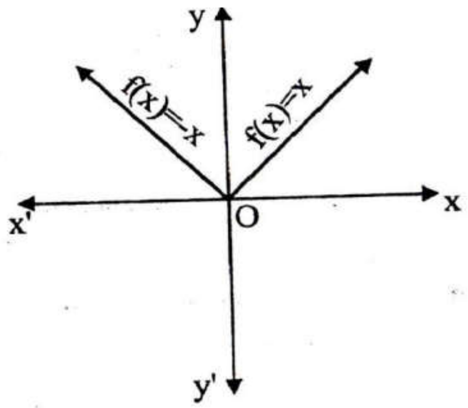
3. Modulus function :-

A function is called the modulus function if defined by

$$f(x) = |x| = \begin{cases} x, & if \quad x \geq 0 \\ -x, & \text{When} \quad x < 0 \end{cases}$$

We can say that modulus of every number is a non-negative real number, so

Domain = *R and Range* = non-negative real numbers.



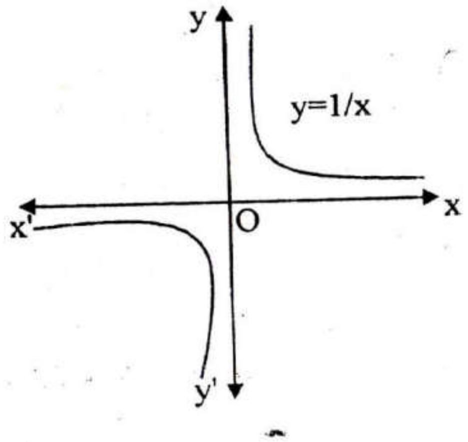
4. Reciprocal function :-

The function defined by $f(x)=\frac{1}{x}$ is called the reciprocal function.

The function is not defined for $x = 0$.

Domain $= R - \{0\}$ and Range

Also, .



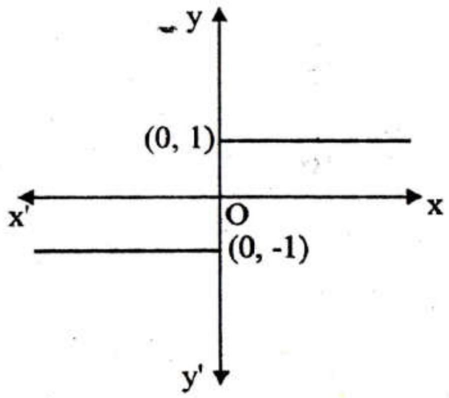
5. Signum function :-

The function is defined by

$$f(x)=\begin{cases}\frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0\end{cases}$$

$$or\ f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

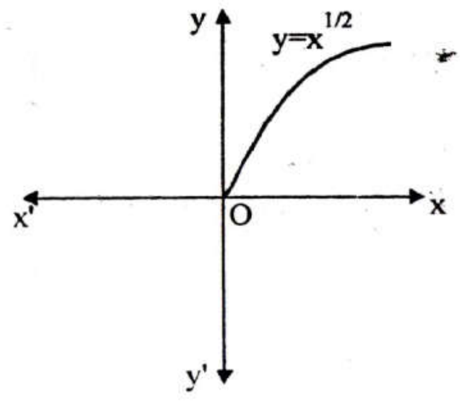
Domain = R and Range = {-1, 0, 1}.



6. Square-root function :-

The function defined by $f(x) = +\sqrt{x}$ is known as square root funtion. As we know that negative numbers do not have real square roots. So, *f(x)* is not defined for negative real number.

Domain = set of all non-negative real numbers $= [0, \infty[$ and Range = set of all non-negative real numbers $= [0, \infty[$.



Q. 7. Differentiating $f(x) = \sin x$

~ol. Here $f(x) = \sin x$ so that $f(x + \delta x) = \sin(x + \delta x)$.

So $\quad f(x+\delta x) - f(x) = \sin(x+\delta x) - \sin x$

The right hand side is the difference of two sine terms. We use the first trignometric identity (above) to write this in an alternative from.

$$\sin(x+\delta x) - \sin x = 2\cos\frac{x+\delta x+x}{2}\sin\frac{\delta x}{2}$$

$$= 2\cos\frac{2x+\delta x}{2}\sin\frac{\delta x}{2}$$

$$= 2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

Then, using the definition of the derivative

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$= \frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$$

The factor of 2 can be moved into the denominator as follows, in order to write this in an alternative form :

$$\frac{dy}{dx} = \frac{\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x/2} \qquad = \cos\left(x+\frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

We now let δx tend to zero. Consider the term $\dfrac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$ and use the result

that $\lim_{\theta \to 0}\dfrac{\sin\theta}{\theta} = 1$ with $\theta = \dfrac{\delta x}{2}$. We see that

$$\lim_{\delta x \to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} = 1$$

Further, $\lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) = \cos x$

So finally, $\frac{dy}{dx} = \cos x$

OR

Q. 7 A $f(x)$ be a function defined by $f(x) = \begin{cases} 4x-5, & \text{if } x \le 1 \\ 2x-\lambda, & \text{if } x > 1 \end{cases}$ Find λ, if limit of function f(x) exists at x = 1.

Sol. : Given funtion is:

$$f(x) = \begin{cases} 4x-5, & \text{if } x \le 1 \\ 2x-\lambda, & \text{if } x > 1 \end{cases}$$

Now,

Left hand limit (LHL) at x = 1

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$$

$$= \lim_{h \to 0} 4(1-h) - 5$$

$$= \lim_{h \to 0} (-1-4h) = -1$$

Right hand limit (RHL) at $x = 1$

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$$

$$= \lim_{h \to 0} 2(1+h) - \lambda$$

$$= (2-\lambda)$$

Since $\lim_{x \to 1} f(x)$ exists, So

LHL = RHL

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$\Rightarrow \quad 1 = 2 - \lambda$

$\Rightarrow \quad \lambda = 1$

Q.8. Prove that the following $f(x)=|x|+|x-1|$ is continuous at x = o, 1 but not differentiable.

Sol. $f(x)=|x|+|x-1|$

$$f(x)=\begin{cases}(-x)-(x-1) & x<o\\ =1-2x & x<o\\ 1 & o\le x<1\\ (x)+(x+1)=2x-1 & x\ge 1\end{cases}$$

$x=o \quad f(0)=1$

$f(o+n)=1 f(o-n)=1-2(o-n)=1+2n$

$\text{RHL}=\lim_{n\to 0} 1 f(o+n)=\lim_{n\to 0}(1)=1$

$\text{LHL}=\lim_{n\to 0} 1 f(o-n)=\lim_{n\to 0}(1+2h)=1$

$f(o)=RHL=LHL$

Hence the function is continuous at x = o

$$Rf'(o)=\lim_{n\to 0}\frac{f(o+n)-f(o)}{h}=\lim_{n\to 0}\left(\frac{1-1}{h}\right)=o$$

$$Lf'(o)=\lim_{n\to 0}\frac{f(o-n)-f(o)}{-h}=\lim_{n\to 0}\frac{1+2n-1}{-h}=-2$$

$Rf'(0)\neq Lf'(0)$.

Hence function is not differentiable at x = 0

Again at $x=1 f(1)=2-1=1$

$f(1+n)=2(1+n)-1=1+2n$ and $f(1-n)=1$

$RHL=\lim_{h\to 0} f(1+n)=\lim_{h\to 0}(1+2n)=-1$

$LHL=\lim_{h\to 0} f(1-n)=\lim_{n\to 0}(1)=1\Rightarrow f(1)=RHL=LHL$

Hence function is continuous at x = 1

$$Rf'(1)=\lim_{h\to 0}\frac{f(1+n)-f(1)}{h}=\lim_{h\to 0}\frac{(1+2n-1)}{h}=2$$

$$Lf'(1)=\lim_{n\to 0}\frac{f(1-n)-f(1)}{h}=\lim_{h\to 0}\frac{1-1}{h}=0$$

$$Rf'(1)\neq Lf'(1)$$

Hence function is not differentiable at x = 1

Q.9. Differinltiate $\sin^{+}x$ by first principle.

Ans. Proof. $x=\sin y$

$$x+\delta x=\sin(y+\delta y)$$

$$\delta x=\sin(y+\delta y)-\sin y$$

$$\delta x=2\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}\qquad\left[\because \sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right.$$

$$\frac{\delta x}{\delta y}=\frac{2\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}}{\delta y}$$

$$\frac{\delta x}{\delta y}=\frac{\frac{\delta y}{2}}{\cos\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)}$$

When $\delta x\to 0$ then $\delta y\to 0$

$$\therefore \delta\lim_{x\to 0}\frac{\delta y}{\delta x}=\delta\lim_{y\to 0}\frac{\frac{\delta y}{2}}{\cos\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}}$$

$$\delta\lim_{x\to 0}\frac{\delta y}{\delta x}=\delta\lim_{y\to 0}\frac{\frac{\delta y}{2}}{\sin\left(\frac{\delta y}{2}\right)}\times\delta\lim_{y\to 0}\frac{1}{\cos\left(y+\frac{\delta y}{2}\right)}$$

$$\frac{dy}{dx}=1\times\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

Derivative of $\cos^{+}x$ by first principle.

$$y=\cos^{+}x$$

$$x=\cos y$$

Again let $x+\delta x=\cos(y+\delta y)$

$$\delta x=2\sin\left(y+\frac{\delta y}{2}\right)\sin\left(-\frac{\delta y}{2}\right)$$

$$=-2\sin\left(y+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)$$

$$\frac{\delta x}{\delta y}=-\frac{\sin\left(1+\frac{\delta y}{2}\right)\sin\left(\frac{\delta y}{2}\right)}{\left(\frac{\delta y}{2}\right)}$$

$$\frac{\delta y}{\delta x}=-\frac{\frac{\delta y}{2}}{\sin\left(y+\frac{\delta y}{2}\right)\sin\frac{\delta y}{2}}$$

when $\delta x\to 0$ then $\delta y\to 0$

$$\delta\lim_{x\to 0}\frac{\delta y}{\delta x}=\delta\lim_{y\to 0}\frac{-1}{\sin\left(y+\frac{\delta y}{2}\right)\frac{\sin\left(\frac{\delta y}{2}\right)}{\left(\frac{\delta y}{2}\right)}}$$

$$\frac{dy}{dx}=\frac{-1}{\sin y\times 1}=\frac{-1}{\sqrt{1-\cos^2 y}}=\frac{-1}{\sqrt{1-x^2}}$$

Q.10. Find the derivative of $\frac{x+1}{x+2}$ w.r.t. x by first principle.

Sol. Let $y=\frac{x+1}{x+2}$

$$y+\delta y=\frac{x+\delta x+1}{x+\delta x+2}$$

$$\delta y=\frac{x+\delta x+2}{x+\delta x+2}-\frac{x-1}{x-2}$$

$$= \frac{(x+\delta x+1)(x+2)-(x+1)(x+\delta x+2)}{(x+\delta x+2)(x+2)}$$

$$\delta y = \frac{x^2 + x\delta x + x + 2x + 2\delta x + 2 - x^2 - x\delta x - 2x - x - \delta x - 2}{(x+\delta x+2)(x+2)}$$

$$\frac{\delta y}{\delta x} = \frac{\delta x}{(x+\delta x+2)(x+2)\delta x} = \frac{1}{(x+\delta x+2)(x+2)}$$

$$\delta x \xrightarrow{\lim} 0 \frac{\delta y}{\delta x} = \delta x \xrightarrow{\lim} 0 \frac{1}{(x+\delta x+2)(x+2)}$$

$$\frac{dy}{dx} = \frac{1}{(x+2)(x+2)} = \frac{1}{(x+2)^2}$$

Unit -3

Very Short Questions

1. **If two matrices contain the same values, but in different locations, are the matrices equal?**

Ans. No, to be equal, two matrices must have the same dimensions, and must have the same values in the same positions.

For two matrices to be equal, they must have

1. The same dimentions.

2. Corresponding elements must be equal

Example

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3\times 2} \neq \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}$$

2. **Are these two matrcies equal?**

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = ? = \begin{pmatrix} 6 & 4 \\ 5 & 2 \\ 1 & 3 \end{pmatrix}$$

Ans. No, they have the same dimensions, but corresponding elements are not equal.

3. **What is:**

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}^{T} = ?$$

Ans. $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2\times 3}^{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}_{3\times 2}$$

4. **What is**

$$\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T = ?$$

Ans. $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$

5. **What is the transpose of I?**

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}^T$$

Ans. $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$

6. **What is $(AB)(B^{-1}A^{-1})$?**

Ans. $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$, by associativity.

$$= A\,1A^{-1} = AA^{-1} = 1$$

7. **What is the rank of the following matrix?**

$$\begin{pmatrix}1 & 2 & 0 & 3\\1 & -2 & 3 & 0\\0 & 0 & 4 & 8\\2 & 4 & 0 & 6\end{pmatrix}$$

Ans. Three. The last row is a multiple of the first.

8. **What is (21)(0.51)?**

Ans. (21) (0.51) = 1

or

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The resulting matrix is 1

9. Is Ax equal to xA?

Ans. No, matrix multiplication is NOT commutative.

10. Perform the following multiplication:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix}_{4\times 4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}_{4\times 1} = ?$$

Ans. $$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & -1 & 2 & 0 \end{pmatrix}_{4\times 4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}_{4\times 1} = \begin{pmatrix} -1 \\ -1 \\ 3 \\ 2 \end{pmatrix}_{4\times 1}$$

11. What dimensions will the following product have:

$A_{4\times 4} B_{4\times 2} = C_{?\times ?}$

Ans. $A_{4\times 4} B_{4\times 2} = C_{4\times 2}$

If two rectangular matrices are put in order so that the inner dimension is the same in each, then the matrices can be multiplied. The result is (in general) a rectangular matrix.

$A_{R\times N} B_{N\times C} = C_{R\times C}$

12. Is it always true that AB = BA?

Ans. No, Even when both products can be formed they are rarely equal.

Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

Is equal to

$$\begin{pmatrix} 5 & -1 \\ 11 & -3 \end{pmatrix}$$

In the matrix

$$A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$$ **, write**

(i) The order of the matrix

(ii)The number of elements.

(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$

Ans. (i) In the given matrix, the number of rows is 3 and the number of columns is 4. Therefore, thr order of the matrix is 3×4.

(ii) Since the order of the matrix is 3×4, there are $3 \times 4 = 12$ elements in it.

(iii) $a_{13} = 19, a_{21} = 35, a_{33} = -5, a_{24} = 12, a_{23} = \frac{5}{2}$.

13. Find the value of x, y and z from the following equation:

$$\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$$

Ans. As the given matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get:

14. Compute the following

$$\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

Ans. $$\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \left(\because \sin^2 x + \cos^2 x = 1\right)$$

15. Compute the indicated products

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Ans. $$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} a(a) + b(b) & a(-b) + b(a) \\ -b(a) + a(b) & -b(-b) + a(a) \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & -ab + ab \\ -ab + ab & b^2 + a^2 \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$$

16. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Ans. (i) Let $A=\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$, then $A^T=\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$.

(ii) Let $A=\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, then $A^T=\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

17. Find the inverse of each of the matrices, if it exists

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Ans. Let $A=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

We know that $A=IA$,

$$\therefore \quad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}A \quad (R_1 \to R_1 - R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}A \quad (R_2 \to R_2 - R_1)$$

$$\therefore A^{-1}=\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

18. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(ii) Let

$$A = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

By Expanding along the first row, we have:

$$|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(1+6) + 4(1+4) + 5(3-2)$$

$$= 3(7) + 4(5) + 5(1)$$

$$= 21 + 20 + 5 = 46$$

19. Find area of the triangle with vertices at the point given in each of the following:

(i) (1, 0), (6, 0), (4, 3)

Ans. (i) The are of the triangle with vertices (1, 0), (6, 0), (4, 3) is given by the relation

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(0-3) - 0(6-4) + 1(18-0)\right]$$

$$= \frac{1}{2}[-3+18] = \frac{15}{2} \text{ square units}$$

20. Write Minors and Cofactors of the elements of the following determinants:

$$\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$$

Ans. The given determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

Minor of element a_{ij} is M_{ij}.

$\therefore M_{11}$ = minor of element $a_{11} = 3$

M_{12} = minor of element $a_{12} = 0$

M_{21} = minor of element $a_{21} = -4$

M_{22} = minor of element $a_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

21. Find adjoint of each of the matrices

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Ans. Let

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

We have,

$A_{11} = 4, A_{12} = -3, A_{21} = -2, A_{22} = 1$

$\therefore adj A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

2. Find the inverse of each of the matrices (if it exists):

$\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

Ans. Let

$$A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$$

we have,

$|A| = 6 + 8 = 14$

Now,

$A_{11} = 3, A_{12} = -4, A_{21} = 2, A_{22} = 2$

$$\therefore adjA = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

23. Examine the consistency of the system of equations

$x + 2y = 2$

$2x + 3y = 3$

Ans. The given system of equations is:

$x + 2y = 2$

$2x + 3y = 3$

The given system of equations can be written in the form of AX = B, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Now,

$|A| = 1(3) - 2(2) = 3 - 4 = -1 \neq 0$

$\therefore$ A is non-singular.

Therefore A^{-1} exists

Hence, the given system of equations is consistent

24. **If** $A=\begin{bmatrix}1 & 1 & -2\\ 2 & 1 & -3\\ 5 & 4 & -9\end{bmatrix}$, find $|A|$.

Ans. Let

$$A=\begin{bmatrix}1 & 1 & -2\\ 2 & 1 & -3\\ 5 & 4 & -9\end{bmatrix}$$

By expanding along the first row, we have:

$$|A|=\begin{vmatrix}1 & -3\\ 4 & -9\end{vmatrix}-1\begin{vmatrix}2 & -3\\ 5 & -9\end{vmatrix}-2\begin{vmatrix}2 & 1\\ 5 & 4\end{vmatrix}$$

$=1(-9+12)-1(-18+15)-2(8-5)$

$=1(3)-1(-3)-2(3)$

$=3+3-6$

$=0$

25. **If** $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$, **then show that** $|2A|=4|A|$

Ans. The given matrix is $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$

$\therefore 2A=2\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}=\begin{bmatrix}2 & 4\\ 8 & 4\end{bmatrix}$

$\therefore L.H.S=|2A|=\begin{bmatrix}2 & 4\\ 8 & 4\end{bmatrix}$

$=2\times 4-4\times 8=8-32=-24$

Now,

$$|A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 1\times 2 - 2\times 4 = 2 - 8 = -6$$

$$\therefore R.H.S = 4|A| = 4\times(-6) = -24$$

$$\therefore L.H.S = R.H.S$$

Short Questions

Q.1. Define Matrices and explain special types of Matric

Ans. A rectangular arrangement of mn numbers, in m rows a columns and enclosed within a bracket is called a matrix shall denote matrices by capital letters as A, B, C, etc:

$$A = \begin{pmatrix} a_{11}a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m\times n}$$

A is a matrix of order $m\times n$, i^{th} row j^{th} column element matrix denoted by a_{ij}

Special Types of Matrices:

Square Matrix

A matrix in which numbers of rows are equal to number o umns is called a square matrix.

$$\begin{pmatrix} 1 & 2 & -5 \\ 3 & 6 & 5 \\ 0 & -1 & 4 \end{pmatrix}$$

Rectangular Matrix

A rectangular matrix is formed by a different number o and columns, and its dimension is noted as: $m\times n$

$$\begin{pmatrix} 1 & 2 & 5 \\ 9 & 1 & 3 \end{pmatrix}$$

Zero Matrix

In a zero matrix, all the elements are zeros.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Diagonal Matrix

In a diagonal matrix, all the elements above and below the diagonal are zeros.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Scalar Matrix

A scalar matrix is a diagonal matrix in which the diagonal elements are equal.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Identity Matrix

An identity matrix is a diagonal matrix in which the diagonal elements are equal to 1.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q.2. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

(i) $A + B$

(ii) $3A - C$

Ans. (i) $A + B = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+3 \\ 3-2 & 2+5 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$

(ii) $3A - C = 3\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3\times 2 & 3\times 4 \\ 3\times 3 & 3\times 2 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 6 & 12 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 6+2 & 12-5 \\ 9-3 & 6-4 \end{bmatrix}$

$= \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$

Q.3. Find x and y, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

Ans. $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

Comparing the corresponding elements of these two matrices, we have:

$2 + y = 5 \Rightarrow y = 3$

$2x + 2 = 8$

$\Rightarrow x = 3$

$\therefore x = 3; y = 3$

4. If $A=\begin{bmatrix} 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(i) (A+B)' = A' + B'

We have:

$$A'=\begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} \text{ and } B=\begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$$

(i) $$A+B=\begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}+\begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}$$

$$\therefore (A+B)'=\begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

$$A'+B'=\begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix}+\begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}=\begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

Hence, we have verified that (A+B)' = A' + B'

5. If (i) $A=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, **then verify that** $A'A=I$

is (i) $A=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$A'A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\alpha)(\cos\alpha)+(-\sin\alpha)(-\sin\alpha) & (\cos\alpha)(\sin\alpha)+(-\sin\alpha)(\cos\alpha) \\ (\sin\alpha)(\cos\alpha)+(\cos\alpha)(-\sin\alpha) & (\sin\alpha)(\sin\alpha)+(\cos\alpha)(\cos\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & \sin\alpha\cos\alpha-\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha-\sin\alpha\cos\alpha & \sin^2\alpha+\cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, we have verified that $A'A = I$

Q.6. Find the inverse of each of the matrices, if it exists

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

Ans. Let

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

We know that $A = IA$

$$\therefore \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \quad (R_2 \to R_2 - 2R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A \left(R_2 \to \frac{1}{5}R_2\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} A \quad (R_1 \to R_1 + R_2)$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Q.7. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

Ans. Answer:C

$(I + A)^3 - 7A = I^3 + A^3 + 3I^2A + 3AI^2 - 7A$

$= I + A^3 + 3A + 3A^2 - 7A$

$= I + A^2.A + 3A + 3A - 7A \qquad [A^2 = A]$

$= I + A.A - A$

$= I + A - A$

$= I + A - A$

$= I$

$\therefore (I + A)^3 - 7A = I$

Q. 8. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

Ans. The given matrix is $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$

$\therefore 2A = 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$

$$L.H.S = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2 \times 4 - 4 \times 8 = 8 - 32 = -24$$

Now,

$$|A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 1 \times 2 - 2 \times 4 = 2 - 8 = -6$$

$$\therefore R.H.S = 4|A| = 4 \times (-6) = -24$$

$$\therefore L.H.S = R.H.S$$

Q. 9. Show that the points

$A(a, b+c), B(b, c+a), C(c, a+b)$ are collinear

Ans. Area of ΔABC is given by the relation:

$$\Delta = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b-a & a-b & 0 \\ c-a & a-c & 0 \end{vmatrix}$$

$(Applying\ R_2 \to R_1$ and $R_3 \to R_3 - R_1)$

$$= \frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= \frac{1}{2}(a-b)(c-a)\begin{vmatrix} a & b+c & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

(Applying $R_3 \to R_2 - R_3 + R_2$)

$= 0$

(All elements of R_3 are 0)

Thus, the area of the triangle formed by points A, B and C is zero.

Hence, the points A, B and C are collinear.

Q.10. Find adjoint of each of the matrices

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Ans. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

We have,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2 + 10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1 + 4 = 5$$

$$A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0 - 2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11$$

$$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5$$

Hence, $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$

Q. 11. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$**, find** A^{-1}**. Using** A^{-1} **solve the sys-tem of equations**

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

Ans. $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$\therefore |A| = 2(-4+4) + 3(-6+4) + 5(3-2) = 0 - 6 + 5 = -1 \neq 0$

Now,

$A_{11} = 0, A_{12} = 2, A_{13} = 1$

$A_{21} = -1, A_{22} = -9, A_{23} = -5$

$A_{31} = 2, A_{32} = 23, A_{33} = 13$

$$A^{-1}=\frac{1}{|A|}(adjA)=-\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}$$

$$=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix} \qquad ...(i)$$

Now, the given system of equations can be written in the form of AX = B, where

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}$$

The solution of the system of equations is given by $X=A^{-1}B$.

$X=A^{-1}B$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix}\begin{bmatrix}11\\ -5\\ -3\end{bmatrix} \qquad \text{[Using (1)]}$$

$$=\begin{bmatrix}0-5+6\\ -22-45+69\\ -11-25+39\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

Hence, x=1, y=2, and z=3.

12. Solve the system of equations using Cramer's Rule.

$x-10y=4$

$2x+y=8$

$$x=\frac{D_x}{D}=\frac{\begin{vmatrix}4 & -10\\ 8 & 1\end{vmatrix}}{\begin{vmatrix}1 & -10\\ 2 & 1\end{vmatrix}}$$

$$=\frac{4(1)-(-10)8}{1(1)-(1-10)2}=\frac{4+80}{1+20}=\frac{84}{21}=4$$

$$y=\frac{D_y}{D}=\frac{\begin{vmatrix}1 & 4\\ 2 & 8\end{vmatrix}}{\begin{vmatrix}1 & -10\\ 2 & 1\end{vmatrix}}$$

$$=\frac{1(8)-2(4)}{1(1)-(-10)2}=\frac{8-8}{1+20}=\frac{0}{21}=0$$

Hence the solution of the system is (4,0).

Q.13. Solve the equations using Cramer's rule.

$1.25x+5y=-20$

$3x-3y=27$

Solving the system of equations using the Cramer's rule, we get

$$a=\frac{\begin{vmatrix}-20 & 5\\ 27 & -3\end{vmatrix}}{\begin{vmatrix}1.25 & 5\\ 3 & -3\end{vmatrix}}=\frac{-20(-3)-5(27)}{1.25(-3)-5(3)}$$

$$a=\frac{60-135}{-3.75-15}=\frac{-75}{-18.75}$$

$a=4$

Usiing Cramer's Rule,

$$b=\frac{\begin{vmatrix}1.25 & -20\\ 3 & 27\end{vmatrix}}{\begin{vmatrix}1.25 & 5\\ 3 & -3\end{vmatrix}}=\frac{1.25(27)-(-20)(3)}{1.25(-3)-5(3)}$$

$$b=\frac{33.75+60}{-3.75-15}=\frac{93.75}{-18.75}$$

$b = -5$

Thus, the solution is $(4, -5)$.

Q. 14. If area of triangle is 35 square units with vertices (2, –6), (5, 4)and (k, 4). Then k is?

Ans. The area of the triangle with vertices (2, –6), (5, 4)and (k, 4)is given by the relation.

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$= \frac{1}{2}[30-6k+20-4k] \quad = \frac{1}{2}[50-10k]$$

$$= 25 - 5k$$

It is given that the area of the triangle is ± 35

Therefore, we have:

$$\Rightarrow 25-5k = \pm 35 \;\Rightarrow 5(5-k) = \pm 35 \;\Rightarrow 5-k = \pm 7$$

When 5 – k = –7, k = 5+7 = 12

When 5 – k = 7, k = 5 – 7 = –2

Hence, k = 12, –2

Q. 15. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$**, verify that**

(i) $(A + A')$ **is a symmetric matrix**

(ii) $(A - A')$ **is a skew symmetric matrix.**

Ans. $A' = \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$

(i) $A+A'=\begin{bmatrix}1 & 5\\6 & 7\end{bmatrix}+\begin{bmatrix}1 & 6\\5 & 7\end{bmatrix}=\begin{bmatrix}2 & 11\\11 & 14\end{bmatrix}$

$\therefore (A+A')=\begin{bmatrix}2 & 11\\11 & 14\end{bmatrix}=A+A'$

Hence, $(A+A')$ is a symmetric matrix.

(ii) $A-A'=\begin{bmatrix}1 & 5\\6 & 7\end{bmatrix}-\begin{bmatrix}1 & 6\\5 & 7\end{bmatrix}=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

$(A-A')'=\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}=-\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}=-(A-A')$

Hence, $(A-A')$ is a skew-symmetric matrix.

Long Questions

Q.1 If A $=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}, B=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}4 & 1 & 2\\0 & 3 & 2\\1 & -2 & 3\end{bmatrix}$

then compute (A+B) and (B-C). Also, verify that A+(B-C)=(A+B)–C

Ans. $A+B=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}+\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}$

$=\begin{bmatrix}1+3 & 2-1 & -3+2\\5+4 & 0+2 & 2+5\\1+2 & -1+0 & 1+3\end{bmatrix}=\begin{bmatrix}4 & 1 & -1\\9 & 2 & 7\\3 & -1 & 4\end{bmatrix}$

$$B-C=\begin{bmatrix}3&-1&2\\4&2&5\\2&0&3\end{bmatrix}-\begin{bmatrix}4&1&2\\0&3&2\\1&-2&3\end{bmatrix}$$

$$=\begin{bmatrix}3-4&-1-1&2-2\\4-0&2-3&5-2\\2-1&0-(-2)&3-3\end{bmatrix}=\begin{bmatrix}-1&-2&0\\4&-1&3\\1&2&0\end{bmatrix}$$

$$A+(B-C)=\begin{bmatrix}1&2&-3\\5&0&2\\1&-1&1\end{bmatrix}+\begin{bmatrix}-1&-2&0\\4&-1&3\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}1+(-1)&2+(-2)&-3+0\\5+4&0+(-1)&2+3\\1+1&-1+2&1+0\end{bmatrix}=\begin{bmatrix}0&0&-3\\9&-1&5\\2&1&1\end{bmatrix}$$

$$(A+B)-C=\begin{bmatrix}4&1&-1\\9&2&7\\3&-1&4\end{bmatrix}-\begin{bmatrix}4&1&2\\0&3&2\\1&-2&3\end{bmatrix}$$

$$=\begin{bmatrix}4-4&1-1&-1-2\\9-0&2-3&7-2\\3-1&-1-(-2)&4-3\end{bmatrix}=\begin{bmatrix}0&0&-3\\9&-1&5\\2&1&1\end{bmatrix}$$

Hence we have verified that A+(B – C)= (A+B) – C

Q.2 Solve the system of equations using Cramer's rule.

$2x + 2y - z = 4$

$x + 2y + z = 14$

$x + 4y + z = 8$

Ans. Step 1 : First we have to find the values of the determinants D, D_x, D_y and D_z.

$$D=\begin{vmatrix}2&2&-1\\1&2&1\\1&4&1\end{vmatrix}$$

$$= 2(2-4)-1(2+4)-1(2+2) = -6$$

$$D_x = \begin{vmatrix} 4 & 2 & -1 \\ 14 & 2 & 1 \\ 8 & 4 & 1 \end{vmatrix}$$

$$= 4(2-4)-2(14-8)-1(56-16) = -60$$

Step 2 : The other determinants D_y and D_z are :

$$D_y = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 14 & 1 \\ 1 & 8 & 1 \end{vmatrix}$$

$$= 2(14-8)-4(1-1)-1(8-14) = 18$$

$$D_z = \begin{vmatrix} 2 & 2 & 4 \\ 1 & 2 & 14 \\ 1 & 4 & 8 \end{vmatrix}$$

$$= 2(16-56)-1(16-16)+(28-8) = -60$$

Step 3 : Apply Cramer's rule

$$x = \frac{D_x}{D} = \frac{-60}{-6} = 10$$

$$y = \frac{D_y}{D} = -\frac{18}{6} = -3$$

$$z = \frac{D_z}{D} = \frac{-60}{-6} = 10$$

Hence the solution of the system is (10, –3, 10)

Q3. Solve the following system of equation using Cramer's Rule.

$$5x - 7y + z = 11,$$

$$6x - 8y - z = 15$$

and $3x + 2y - 6z = 7$

Sol. The given system of equation is

$$5x-7y+z=11$$
$$6x-8y-z=15$$
$$3x+2y-6z=7$$

$$\therefore D=\begin{vmatrix}5 & -7 & 1\\ 6 & -8 & -1\\ 3 & 2 & -6\end{vmatrix}=5(48+2)+7(-36+3)+1(12+24)$$
$$=250-231+36$$
$$=55$$

$$D_1=\begin{vmatrix}11 & -7 & 1\\ 15 & -8 & -1\\ 7 & 2 & -6\end{vmatrix}=11(48+2)+7(-90+7)+1(30+56)$$
$$=550-581+86$$
$$=55$$

$$D_2=\begin{vmatrix}5 & 11 & 1\\ 6 & 15 & -1\\ 3 & 7 & -6\end{vmatrix}=5(-90+7)-11(-36+3)+1(42-45)$$
$$=-415+363-3$$
$$=-55$$

$$D_3=\begin{vmatrix}5 & -7 & 11\\ 6 & -8 & 15\\ 3 & 2 & 7\end{vmatrix}=5(-56-30)+7(42-45)+11(12+24)$$
$$=-430-21+396$$
$$=-55$$

So by Cramer's Rule $x=\dfrac{D_1}{D}=\dfrac{55}{55}=1$

$y=\dfrac{D_2}{D}=\dfrac{-55}{55}=-1$

and $z=\dfrac{D_3}{D}=\dfrac{-55}{55}=-1$

Q.4. $A=\begin{bmatrix}4 & 1 & 5\\0 & 7 & 3\\0 & 0 & 5\end{bmatrix}$

Find the eigen value and eigen vector of above.

Sol. Characteristic equation:

$$(4+\lambda)(7-\lambda)(5-\lambda)=0$$

$$\Rightarrow \lambda=4,7,5$$

- **Eigen Vectors:**

The eigen vector X of a martix A, corresponding to eigen valu λ is given by the non-zero solutions of the equation $[A-\lambda I]X=0$

Eg: $A=\begin{bmatrix}4 & 1 & 5\\0 & 7 & 3\\0 & 0 & 5\end{bmatrix}$

Characteristic equation:

$$(4/\lambda)(7-\lambda)(5-\lambda)=0$$

$$\Rightarrow \lambda=4,7,5$$

Eigen Vector, corresponding to eigen value 4:

$$[A-\lambda I]\times=0$$

$$\Rightarrow\left(\begin{bmatrix}4 & 1 & 5\\0 & 7 & 3\\0 & 0 & 5\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}0 & 1 & 5\\0 & 3 & 3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow x_2+5x_3=0$$

$$3x_2 + 3x_3 = 0$$

$$x_3 = 0$$

$$\Rightarrow x_3 = 0$$

$$\Rightarrow x_2 = 0$$

Let

$$x_1 = k$$

So Eigen vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$$

Eigen Vector, corresponding to Eigen value 7:

$$[A - \lambda I] \times = 0$$

$$\Rightarrow \left(\begin{bmatrix} 4 & 1 & 5 \\ 0 & 7 & 3 \\ 0 & 0 & 5 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3x_1 + x_2 + 5x_3 = 0$$

$$3x_3 = 0$$

$$-2x_3 = 0$$

$$\Rightarrow x_3 = 0$$

$$\Rightarrow -3x_1 + x_2 = 0$$

Say, $x_1 = k$

$\Rightarrow x_2 = 3k$

So eigen vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 3k \\ 0 \end{bmatrix}$$

Q. 5. Solve the equation by Cramer's Rule.

$$x + y + z = 6$$

$$x - y + z = 2$$

$$2x + y - z = 1$$

Sol.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 6$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 6$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 12$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 18$$

So, $x = \dfrac{\Delta_1}{\Delta} = \dfrac{6}{6} = 1$

$$y = \frac{\Delta_2}{\Delta} = \frac{12}{6} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{18}{6} = 3$$

Q6. Find the value of x and y through inverse method.

5x + 2y = 3

3x + 2y = 5

Sol. In Matrix form:

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

A.X = B

$$A^{-1} = \frac{AdjA}{|A|}$$

$$AdjA = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$|A| = 4$$

So, $$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\Rightarrow x = -1, y = 4$$

Q7. Find the inverse of square matrix.

$$A = \begin{bmatrix} 1 & -4 \\ 7 & 9 \end{bmatrix}$$

Sol. $|A| = \begin{vmatrix} 1 & -4 \\ 7 & 9 \end{vmatrix}$

$= 1 \times 9 - (-4) \times 7 = 37$

$C = \begin{bmatrix} 9 & -7 \\ 4 & 1 \end{bmatrix}$

$adj A = C^T$

$adj A = \begin{bmatrix} 9 & 4 \\ -7 & 1 \end{bmatrix}$

So, $A^{-1} = \frac{1}{37}\begin{bmatrix} 9 & 4 \\ -7 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{9}{37} & \frac{4}{37} \\ -\frac{7}{37} & \frac{1}{37} \end{bmatrix}$

Eg: $A = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 3 & 6 \\ 1 & 4 & 9 \end{bmatrix}$

$|A| = \begin{vmatrix} 4 & 3 & 5 \\ 2 & 3 & 6 \\ 1 & 4 & 9 \end{vmatrix}$

$= 1$

$C = \begin{bmatrix} 3 & -12 & 5 \\ -7 & 31 & -13 \\ -7 & -14 & 6 \end{bmatrix}$

$$adjA = C^T = \begin{bmatrix} 3 & -7 & -7 \\ -12 & 31 & -14 \\ 5 & -13 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -7 & -7 \\ -12 & 31 & -14 \\ 5 & -13 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -7 & -7 \\ -12 & 31 & -14 \\ 5 & -13 & 6 \end{bmatrix}$$

Q8. **If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19}A$**

Sol. We have

$$|A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19 \neq 0$$

Therefore, A is invertible. Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$ then,

$C_{11} = -2, C_{12} = -5, C_{21} = -3$ and $C_{22} = 2$

$$\therefore adjA = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}adjA = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$= \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19}A$$

Q.9. **If A and B are two invertible matrices, then find inverse of matrix AB.**

Sol. $\because$ A and B are invertible

So A^{-1} and B^{-1} are both exists.

Now, $(AB)\left(B^{-1}A^{-1}\right)=A\left(BB^{-1}\right)A^{-1}$ $\quad \left\{BB^{-1}=I\right\}$

$=AIA^{-1}$

$=\left(AA^{-1}\right)=I$ $\quad \{\because AI=A\}$

and $\left(B^{-1}A^{-1}\right)(AB)=B^{-1}\left(A^{-1}A\right)B$

$=B^{-1}IB$

$=B^{-1}B=I$

$(AB)\left(B^{-1}A^{-1}\right)=I\left(B^{-1}A^{-1}\right)(AB)$

$\Rightarrow AB$ is invertible.

$\Rightarrow (AB)^{-1}=B^{-1}A^{-1}$

Q10. (i) Show that the points $(a,b+c),(b,c+a)$ and $(c,a+b)$ are collinear.

Sol. We have

$$\Delta=\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

[Applying $c_2 \to c_2+c_1$]

$$\begin{vmatrix} a & a+b+c & 1 \\ b & b+c+a & 1 \\ c & c+a+b & 1 \end{vmatrix}$$

$$\Rightarrow \Delta=(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$$ taking $(a+b+c)$ common from c_2

$\Delta=(a+b+c)\times 0=0$

[$\because c_2$ and c_3 are identical]

Hence, the given points are collinear.

Q10. (ii) If the two points $(a_1, b_1), (a_2, b_2)$ and $(a_1 + a_2, b_1 + b_2)$ are collinear, show that $a_1 b_2 = a_2 b_1$

Sol. The given points are collinear

$$\therefore \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_1 + a_2 & b_1 + b_2 & 1 \end{vmatrix} = 0$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 - a_1 & b_2 - b_1 & 0 \\ a_2 & b_2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 - a_1 & b_2 - b_1 \end{vmatrix} = 0$$ [expanding along c_3]

$$\Rightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$ [applying $R_2 \to R_2 + R_1$]

$$\Rightarrow a_1 b_2 - a_2 b_1 = 0$$

$$\Rightarrow a_1 b_2 = a_2 b_1$$

❑❑❑

UNIT-IV

Co-ordinate Geometry

Very Short Questions

Q1. What is the standard form of a quadratic equation?

Sol.

$$ax^2 + bx + c = 0$$

The quadratic is on the left. 0 is on the right.

Q2. How many roots has a quadratic?

Sol. Always two

Q3. Find the value of the discriminant of the equation, and tell how many real and distinct roots the equation has.

$x^2 - 6x + 5 = 0$

Sol. The expression for the discriminant is $b^2 - 4ac$

Here, a =1, b =-6 and c=5.

Evaluate the expression for the discriminant.

$(-6)^2 - 4(1)(5)$

$= 36 - 20 = 16$

Since the discriminant is positive, there are two real number roots.

Q4. $9m^2 + 24m + 16 = 0$

Sol. The expression for the discriminant is

Here, a=9, b =24, and c=16.

Evaluate the expression for the discriminant.

$= 576 - 576 = 0$

Since the discriminant is zero, there is one real number,root

Q5. $3p^2 - p + 2 = 0$

Sol. The expression for the discriminant is

$b^2 - 4ac$

Here, a =3, b =-1 and c=2.

Evaluate the expression for the discriminant.

(-1)2 - 4(3)(2)

= 1 - 24

= -23

Since the discriminant is negative, there are no real number roots.

Q6. Write an equation using the formula for the discriminant.

Sol. $(24)^2 - 4(4)(k) = 0$

$576 - 16k = 0$

$-16k = -576$

$k = 36$

Q7. Find the quadratic equation whose roots are 5 and 3.

Sol. Given that, roots of the equation are 5 and 3.

Therefore, the required quadratic equation is

$x^2 - (a+b)x + (a*b) = 0$

Substituting the values of roots in the above equation we get the required equation.

$x^2 - (5+3)x + (5*3) = 0$

$x^2 - 8x + 15 = 0$

Q8. Find the solution for following quadratic equation:

$x^2 - 5x = 0$

Sol. $x(x-5) = 0$

$\therefore = 0$ or $x - 5 = 0$

$x = 0$ or $x = 5$

Q9. $x^2 - 4 = 0$

Sol. $x^2 - 4 = 0$

$b^2 - 4ac$

Here, a =3, b =-1 and c=2.

Evaluate the expression for the discriminant.

(-1)2 - 4(3)(2)

= 1 - 24

= -23

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$x^2 - 5x = 0$

Sol. $x(x-5) = 0$

$. = 0$ or $x - 5 = 0$

$x = 0$ or $x = 5$

Q9. $x^2 - 4 = 0$

Sol. $x^2 - 4 = 0$

$$(x-2)(x+2)=0$$

$$x-2=0 \text{ or } x+2=0$$

$$x=2 \text{ or } x=-2$$

Q10. $(x+2)(x+3)=12$

Sol. $(x+2)(x+3)=12$

$$x^2+5x+6=12$$

$$x^2+5x-6=0$$

$$(x+6)(x-1)=0$$

$$x+6=0 \text{ or } x-1=0$$

$$x=-6 \text{ or } x=1$$

Q11. $x^2-3=2x$

Sol. $x^2-3=2x$

$$x^2-2x-3=0$$

$$(x-3)(x+1)=0$$

$$x-3=0 \text{ or } x+1=0$$

$$x=3 \text{ or } x=-1$$

Q12. $x^2-4=0$

Sol. $x^2=4$

$$\sqrt{x^2}=\pm\sqrt{4}$$

$$x=\pm 2$$

Q13. Find the coordinates (x,y) of the midpoint of the segment that connects the points (3,-4) and (-8,6).

Sol. [((x1+x2)/2), ((y1+y2)/2)] = [((3+-4)/2), ((-8+6)/2)] = (-.5,-1)

Q14. Write the equation of the line that passes through the point (2,-3) and has a slope of (1/2). Use slope intercept form.

Sol. Write the general equation used for slope intercept form

$$Y = mx + c$$

Now put the given information.

-3 = (1/2)2 =c

-3 = 1 + c

-4 = c

Now all you have to do is write the equation of the line in gradient-intercept form:

y = 5x - 4

Q15. Find the distance between (-2,3) and (8,-1).

Sol. Distance between two points

D = sqrt[(x2 -x1)^2 + (y2 -y1)^2]

D = sqrt[(8 -(-2))^2 + (-1-3)^2]

D = 2(sqrt(29))

Q16. What is the slope in equation y - 3 = 4(x -2)

Sol. Compare the equation with slope form

y - y1 = m(x - x1)

we can see right away that the slope is 4 because the 4 is in the location of the m.

Q17. Find the equation of a straight line with gradient 4 and y-intercept -1.

Sol. gradient (m) = 4

y - intercept (c) = -1

Equations of a Straight Line Formula: y = mx + c

y = 4x -1

Q18. Find area of triangle whose corners are (a,0) , (0,b) and (x,y).

Sol. $$\text{Area} = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

$$= \frac{1}{2}\left[a(b - y) + 0 + x(0 - b)\right]$$

$$= \frac{1}{2}\left[b(a - x) - ay\right]$$

Q19. Locus of the point for which distance from X-axis is equal to the distance from Y-axis.

Sol. Let the coordinates of the point are (h,k)

By the given condition: h = k

So locus of P: x = y

Q20. Locus of the point for which distance from (3,1) and distance from (-1,2) is always equal.

Sol. Let the coordinates of the point be P(h,k)

Distance of P from (3,1) = Distance of P from (-1,2)

$$\Rightarrow \sqrt{(h-3)^3+(k-1)^2} = \sqrt{(h+1)^2+(k-2)^2}$$

$$= h^2-6h+9+k^2-2k+1 = h^2+2h+1+k^2-4k+4$$

$$\Rightarrow 8h-2k-5=0$$

So locus $P:(8x-2y-5=0)$

Q21. Find the sum and product of the root of the equation $x^2-x-12=0$

Sol. $a=1, b=-1, c=-12$

$\alpha+\beta = \frac{-b}{a} = 1$ $\alpha\beta = \frac{c}{a} = -12$

Q22. Find the quadratic equations whose roots are 8,-3.

Sol. Sum = 8+(-3) = 5

Product = 8 X -3 = -24

Q23. Find the quadratic equation, whose one root is $3+\sqrt{2}$

Sol. Since $3+\sqrt{2}$ is real and its irrational. So other root be its conjugate $3-\sqrt{2}$.

$\alpha+\beta=6$

$\alpha\beta=7$

So quadratic equation $x^2-6x+7=0$

Q24. Find distance between $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\phi, a\sin\phi)$.

Sol. $PQ = \sqrt{(a\cos\phi - a\cos\alpha)^2 + (a\sin\phi - a\sin\alpha)^2}$

$= a\sqrt{(\cos\phi - \cos\alpha)^2 + (\sin\phi - \sin\alpha)^2}$

$= a\sqrt{\left(2\sin\frac{\phi-\alpha}{2}\sin\frac{\alpha-\phi}{2}\right)^2 + \left(2\cos\frac{\phi+\alpha}{2}\sin\frac{\phi-\alpha}{2}\right)^2}$

$= 2a\sin\frac{\alpha-\phi}{2}\sqrt{\sin^2\frac{\alpha+\phi}{2} + \cos^2\frac{\phi+\alpha}{2}}$

$= 2a\sin\frac{\alpha-\phi}{2}$

Q25. Show that points (1,1), (-2,7) and (3,-3) are on a straight line.

Sol. Let $A(1,1), B(-2,7), C(+3,-3)$

$AB = \sqrt{(-2-1)^2 + (7-1)^2}$

$AB = 3\sqrt{5}$

$BC = \sqrt{(3+2)^2 + (-3-7)^2} = 5\sqrt{5}$

$AC = \sqrt{(3-1)^2 + (-3-1)^2} = 2\sqrt{5}$

$\therefore AB + AC = 3\sqrt{5} + 2\sqrt{5}$

$= 5\sqrt{5}$

$= BC$

So A, B, C lies on a straight line.

Q26. The line segment connecting point A(1,-2) and B(4,7), inner partitioned at point C(2,1) in which ratio?

Sol. $x = \dfrac{mx_2 + nx_1}{m+n}$

$\Rightarrow 2 = \dfrac{4m+n}{m+n}$

$\Rightarrow 2m + 2n = 4m + n$

$\Rightarrow n = 2m \Rightarrow \dfrac{m}{n} = \dfrac{1}{2}$

Q27. Equations of a Straight Line Formula: y - y1=m(x - x1) or y = mx + b.

1. Find the equation of a straight line

(a) with gradient 4 and y-intercept -1

(b) passing through the origin with gradient -3

(c) with gradient 4 and x-intercept -5

(d) passing through (2,5) and (-1,1)

(e) passing through (0,1) and (-4,-2)

(f) parallel to the x-axis and passing through (2,3)

(g) parallel to the y-axis and passing through (-1,2)

(h) with gradient -2 and passing through the midpoint of (5,-2) and (-3,4)

Sol. **(a) y = 4x –1**

(b) y = -3x

(c) y = 4x + 20

(d) 4x - 3y + 7 = 0

(e) 3x -4y +4 = 0

(f) y = 3

(g) x = -1

(h) 2x + y -3 = 0

Q28. Find the equation of a circle whose centre is (3,-2) and which passes through the intersection of the line $5x + 7y = 3$ **and** $2x + 3y = 7$

Sol. Given lines are

$5x+7y-3=0$ (1)

$2x-3y-7=0$ (2)

Solving (1) and (2) simultaneously, we get

$x=2, y=-1$

The point of intersection, say P of given lines is (2,-1).

Since the centre of the circle is C(3,-2) and it passes through the point P(2,-1)

Its radius = $CP=\sqrt{(2-3)^2+(-1+2)^2}$

$=\sqrt{1+1}=\sqrt{2}$

$\therefore$ The equation of the circle is

$(x-3)^2+(y+2)^2=\left(\sqrt{2}\right)^2$

$x^2+y^2-6x+4y+11=0$

Q29. Find the equation of a circle whose centre is (-2,3) and radius is 4.

Sol. Since the centre of the circle is (-2,3) and its radius is 4, therefore, the equation of the circle is

$(x-(-2))^2+(y-3)^2=4^2$

$(x+2)^2+(y-3)^2=16$

$x^2+4x+4+y^2-6y+9=16$

$x^2+y^2+4x-6y-3=0$

Q30. Find the coordinate of focus, equation of directrix and the length of latus rectum of the parabola represented by the equation

$x^2=-16y$

Sol. The given equation is $x^2=-16y$

Compare with $x^2=-4ay$

$\Rightarrow 4a=16 \quad \Rightarrow a=4$

The focus is (0,-4) and equation of directrix is $y-4=0$.

The length of latus rectum $=4a=16$

Q31. Find the equation of parabola with vertex at (0,0) and focus at (-2,0).

Sol. Since the focus of the parabola is F(-2,0) which lies on X-axis, the X-axis is the axis of parabola.

Also, the vertex of parabola is at O(0,0)

Therefore the parabola is

$y^2=-4ax$ with $a=2$

Hence required equation of parabola is

$y^2=-4\times 2x$ i.e. $y^2=-8x$

Q32. Find the equation of the line joining the point (2,-9) and the point of intersection of lines $2x+5y-8=0$ and $3x-4y-35=0$

Sol. Solve the equations $2x+5y-8=0$

$$3x-4y-35=0$$

We get $x=9, y=2$

equation of the line joining the given point (2,-9) and the point of intersection (9,-2) is

$$y-(-9)=\frac{(-2)-(-9)}{9-2}(x-2)$$

$x-y-11=0$

which is the required equation.

Q33. Prove that the lines $2x-y+9=0$ and $x+2y-7=0$ are perpendicular to each other.

Sol. The slope of the line $2x-y+9=0$ is

$$m_1=\frac{\text{coff. of x}}{\text{coff. of y}}=\frac{-2}{(-1)}=2 \qquad (1)$$

similarly, slope of the other line $x+2y-7=0$ is

$$m_2 = -\frac{\text{coff. of x}}{\text{coff. of y}} = \frac{-1}{2} \qquad (2)$$

We know that if two lines are perpendicular if product of their slope is -1.

Here $m_1 \times m_2 = 2\times\left(-\frac{1}{2}\right) = -1$

Hence the given lines are mutually perpendicular.

Q34. Prove that the lines $2x-y+9=0$ and $4x-2y-8=0$ are parallel.

Sol. Equation of the first line is $2x-y+9=0$

whose slope is

$$m_1 = -\frac{\text{coff. of x}}{\text{coff. of y}} = 2 \qquad (1)$$

Similarly, equation of the other line is $4x-2y-8=0$

whose slope is

$$m_2 = -\frac{\text{coff. of x}}{\text{coff. of y}} = 2 \qquad (2)$$

Hence $m_1 = m_2$

Short Questions

Q1. Find the solution for following quadratic equation:

$x^2-4x-8=0$

Sol. $x^2-4x=8$

$x^2-4x+4=8+4$

$x^2-4x+4=12$

$(x-2)^2=12$

$$(x-2)^2 = 12$$

$$\sqrt{(x-2)^2} = \pm\sqrt{12}$$

$$x-2 = \pm\sqrt{4.3}$$

$$x-2 = \pm 2\sqrt{3}$$

$$x = 2 \pm 2\sqrt{3}$$

Q2. $(x-2)^2 - 12 = 0$

Sol. $(x-2)^2 = 12$

$$\sqrt{(x-2)^2} = \pm\sqrt{12}$$

$$x-2 = \pm\sqrt{4.3}$$

$$x-2 = \pm 2\sqrt{3}$$

$$x = 2 \pm 2\sqrt{3}$$

Then the solution is $x = 2 \pm 2\sqrt{3}$

Q3. Use the quadratic formula to solve this quadratic equation:

$3x^2 + 5x - 8 = 0$

Sol. We have a =3, b = 5, c =-8

Therefore, according to the formula:

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 3 \cdot (-8)}}{2 \cdot 3}$$

$$= \frac{-5 \pm \sqrt{25+96}}{6}$$

$$= \frac{-5 \pm \sqrt{121}}{6}$$

$$= \frac{-5 \pm 11}{6}$$

That is,

$$x = \frac{-5+11}{6} \text{ or } \frac{-5-11}{6}$$

$$= \frac{6}{6} \text{ or } \frac{-16}{6}$$

$$= 1 \text{ or } \frac{-8}{3}.$$

Q4. Find equation of line which makes intercepts -3 and 2 on X-axis and y-axis respectively.

Sol. $\frac{x}{a}+\frac{y}{b}=1$

$\Rightarrow \frac{x}{-3}+\frac{y}{2}=1$

$=2x-3y+6=0$ [Parth publication, page 84]

Q5. Find the equation of line passing through (-2,3) and with slope -4.

Sol. Line equation passing through with slope m is:

$y-y_1=m(x-x_1)$

$y-3=-4(x+2)$

$4x+y+5=0$

Q6. Find the equation of line whose perpendicular distance from the origin is 5 and the angle which the normal makes with X-axis in positive direction is 30 .

Sol. $\phi=5$

$\alpha=30°$

So equation line: $x\cos\alpha + y\sin\alpha = p$

$x\cos 30° + y\sin 30° = 5$

$x\left(\frac{\sqrt{3}}{2}\right)+y\left(\frac{1}{2}\right)=5$

$[\because \cos 30^\circ = \frac{\sqrt{3}}{2}, \sin 30^\circ = \frac{1}{2}]$

$\sqrt{3}x + y = 10$

Q7. Divide 16 into two parts such that twice the square of the large part exceeds the square of the smaller by 164.

Sol. Let the larger part be x. Then the smaller part = 16-x.

According to question,

$2x^2 = (16-x)^2 + 164$

$\Rightarrow 2x^2 - (16-x)^2 - 164 = 0$

$\Rightarrow x^2 + 32x - 420 = 0$

$\Rightarrow (x+42)(x-10) = 0$

$x = -42$ or $x = 10$

$x = 10$ Ans

Q8. Find the area of quadrilateral whose vertices are A(1,1), B(7,-3), C(12,2) and D(7,21)

Sol. Area of quadrilateral ABCD = $|\text{Area of } \Delta ABC| + |\text{Area of } \Delta ACD|$

Area of $\Delta ABC = \frac{1}{2}\left|\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_3)\right]\right|$

$= \frac{1}{2}\left|[1(-3-2) + 7(2-1) + 12(1+3)\right|$

$= \frac{1}{2}\left|[-5+7+48]\right|$

$= \frac{1}{2}|50|$

$= 25$ sq. unit

Q9. Prove that lines $3x - 4y = 11$ and $4x + 3y = 7$ ar perpendicular to each other.

Sol. $a_1 = 3 \quad b_1 = -4 \quad c_1 = -11$

$a_2 = 4 \quad b_2 = 3 \quad c_2 = -7$

These lines are peprpendicular if $a_1 a_2 + b_1 b_2 = 0$

So, $a_1 a_2 + b_1 b_2$

$= 3 \times 4 + (-4) \times 3$

$= 0$

Q10. Find the equation of circle, whose centre is (2,3) and passes through (5,7).

Sol. radius of circle = distance between centre and point on circle. Centre C(2,3), Points P(5,7)

So, $r = CP = \sqrt{(5-2)^2 + (7-3)^2} = 5$

Equation of circle $(x-h)^2 + (y-k)^2 = r^2$

$(x-2)^2 + (y-3)^2 = 5^2$

$x^2 + y^2 - 4x - 6y - 12 = 0$

Q11. Prove that lines $3x - 4y = 11$ and $4x + 3y = 7$ are perpendicular to each other.

Sol. In general form $a_1 = 3 \quad b_1 = -4 \quad c_1 = -11$

$a_2 = 4 \quad b_2 = 3 \quad c_2 = -7$

These lines be perpendicular if $a_1 a_2 + b_1 b_2 = 0$

So, $a_1 a_2 + b_1 b_2$

$= 3 \times 4 + (-4) \times 3$

$= 0$

So these lines are perpendicular.

Q12. Find centre and radius of the circle $x^2 + y^2 - 4x + 6y - 5 = 0$.

Sol. It is a general form of the circle.

$g = -2$

$f = 3$

$c = -5$

centre $(h,k) = (-g,-f)$

radius $c = h^2 + k^2 - r^2$

$r^2 = h^2 + k^2 - c$

$= 4 + 9 - (-5)$

$r^2 = 18$

$r = \sqrt{18} = 3\sqrt{2}$

Q13. What is (12,5) in polar coordinates ?

Sol. By pythagorous

$r^2 = 12^2 + 5^2$

$r = \sqrt{12^2 + 5^2}$

$r = \sqrt{144 + 25}$

$r = \sqrt{169} = 13$

By tangent function

$\tan\theta = \frac{5}{12}$

$\theta = \tan^{-1}\left(\frac{5}{12}\right)$

$\boxed{\theta = \tan^{-1}\frac{y}{x}}$

formula

Q14. Change the (3,4) to polar co-ordinate.

Sol. Using property

$r^2 = 3^2 + 4^2$

$r^2 = 9 + 16$

$r = \sqrt{9 + 16}$

By tangent function

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\frac{4}{3}$

$\boxed{\theta = \tan^{-1}\frac{4}{3}}$

$r = \sqrt{25} = 5$

Q15. Convert $2x - 5x^3 = 1 + xy$ into polar coordinates

Sol. Replace x by $r\cos\theta$

Replace y by $r\sin\theta$

$$2(r\cos\theta) - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta)$$

$$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

Q16. Convert $r = -8\cos\theta$ into Cartesian coordinate.

Sol. $r^2 = -8r\cos\theta$

$$x^2 + y^2 = -8x$$

Q17. Find sum and products of roots of following equation $4x^2 + 3x + 7 = 0$.

Sol. The sum of roots $= \alpha + \beta = \dfrac{-\text{coefficient of } x}{\text{coefficient of } x^2} = \dfrac{-3}{4}$

Product of roots $= \alpha\beta = \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{7}{4}$

$$\therefore \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-3/4}{7/4} = \frac{-3}{7}$$

Long Questions

Q1. The sum of numbers is 9. The squares of the numbers is 41.

Find the numbers.

These are quadratic simultaneous equations.

Sol. Let the numbers be x and y.

$$x + y = 9$$

$$x^2 + y^2 = 41$$

From the first equation, $y = (9 - x)$

Now substitute this in the second equation.

$$x^2 + (9 - x)^2 = 41$$

$$x^2 + 81 - 18x + x^2 = 41$$

$$2x^2 - 16x + 81 = 41$$

$$2x^2 - 16x + 40 = 0$$

$$x^2 - 8x + 20 = 0$$

$$(x - 5)(x - 4) = 0$$

$$(x - 5) = 0 \text{ or } (x - 4) = 0$$

$x = 5$ or $x = 4$

Substitute this in the first equation, $y = 5$ or 4

The numbers are 5 and 4.

Q2. The three sides of a right angled triangle are ***x*, *x*+1 and 5. Find *x* and the area, if the longest side is 5.**

Sol. The hypotenuse = 5.

$$x^2 + (x + 1)^2 = 5^2 \qquad \text{(Pythagoras' Theorem)}$$

$$x^2 + x^2 + 2x + 1 = 25$$

$$-25 = \; > x^2 + x^2 + 2x - 24 = 0$$

$$(x + 6)(x - 4) = 0$$

$$(x + 6) = 0 \text{ or } (x - 4) = 0$$

$x = -6$ or $x = 4$

$x = 4;$

Area $= 0.5 \times 3 \times 4 = 6\text{cm}^2$

Q3. The sum of two numbers is 27 and their product is 50 Find the numbers.

Sol. Let one number be *x*. Then the other number is 50/*x*.

$x + 50 / x = 27$

$Xx = > x^2 + 50 = 27x$

$-27x = > x^2 - 27x + 50 = 0$

$(x-25)(x-2) = 0$

$(x-25) = 0$ or $(x-2) = 0$

$x = 25$ or $x = 2$.

04. Find the equation of a straight line through the point (-1,3) with slope 2.

Sol. The slope of the line must be the same between any two points, e.g. between (x,y) and (-1,3) so that

$$\frac{y-3}{x-(-1)} = 2$$

$$y-3 = 2(x+1)$$

$$y = 2(x+1)+3$$

$$y = 2x+5$$

5. Find the equation of a line through the point (1,2) and (3,1). What is its slope? What is its y intercept?

Sol. We first find the slope of the line by finding the ratio of the change in y over change in x. Thus

$$m = \frac{(2-1)}{(1-3)} = -\frac{1}{2}$$

Now that we have the slope, the problem is similar to that in Example 1, so we use the same method to conclude that

$$\frac{(y-2)}{(x-1)} = -\frac{1}{2}$$

Rearranging leads to

$$y = -\frac{x}{2} + \frac{5}{2}$$

Thus the y intercept is 5/2 and the slope is -1/2.

Q6. What is the equation of a line that passes through the point (1,1) and is perpendicular to the line $y=-2x+2$? Where do the two lines intersect?

Sol. Recall that the two lines are perpendicular if their slopes are negative reciprocals of one another. Thus, the line $y=mx+b$ would be perpendicular to any line whose slope is $-1/m$.

The slope of the original line is -2. Thus the slope of the second line pependicular to it would be -1/2. We also require that this second line go through the point (1,1). Now we reduced the problem to the type of calculation that we have already did in example 1. We find that points on the line satisfy

$$\frac{(y-1)}{(x-1)}=\frac{1}{2}$$

After rearranging, we find that the required equation is

$$y=\frac{x}{2}+\frac{1}{2}$$

In the second part of this problem we are asked to find the point(s) at which the two lines intersect. First recall that the two lines either intersect at exaclty one point, or else they intersect all along their length (if they correspond to the same line!). In any case, at points of intersection, both equations are satisfied simultaneously. Thus, we look for points (x,y) such that

$$y=-2x+2$$

$$y=\frac{x}{2}\mp\frac{1}{2}$$

The above two equations imply that

$$-2x+2=\frac{x}{2}+\frac{1}{2}$$

$$-2x-\frac{x}{2}=\frac{1}{2}-2$$

$$-\frac{5}{2}x = -\frac{3}{2}$$

$$x = \frac{3}{5}$$

Thus, the x-coordinate of the point of intersection is found. The y value can be obtained by using either of the two equations,and simply plugging in this value of x.

We find that

$$y = -2x + 2 = -2\left(\frac{3}{5}\right) + 2 = \frac{4}{5}$$

The answer is that the point of intersection is $(3/5, 4/5)$

Q15. Find the equation of the tangent line to the circle

$$x^2 + y^2 = 4$$

at the point in the first quadrant whose x coordinate is $x = 1$.

Sol. The equation given in this problem describes a circle with centre at (0,0) and radius 2. Further, note that if the x coordinate of the point on the circle is $x=1$, then the y coordinate must be given by

$$y^2 = 4 - x^2 = 4 - 1 = 3$$

$$y = \pm\sqrt{(3)}$$

We observe that there are two values of y that correspond to the x coordinate $x=1$, but since we are interested in a point in the first quadrant, we select the positive value. This means that the point of the circle at which we want a tangent line has the coordinates $(1, \sqrt{(3)})$. Thus, the radius vector is a line connecting (0,0) and $(1, \sqrt{(3)})$, which means that its slope is

$$m = \frac{\sqrt{3}}{1} = \sqrt{(3)}$$

The tangent line, by what we have said above, is pependicular to this radius vector, and has a slope the negative reciprocal of tl above, namely

$$m = \frac{-1}{\sqrt{3}}$$

We now have all the information required to compute the equatic of the tangent line: namely its slope, $m = \frac{-1}{\sqrt{3}}$ and a point throug which it passes, $(1, \sqrt{(3)})$. We find after some simple algeb that the equation of the tangent line is:

$$y = \frac{-1}{\sqrt{3}}(x-4)$$

Q8. Find the vertex, the focus, the axis of symmetry and th directrix of the parabola defined by the equation

$2y^2 + 8y + x + 1 = 0$

Sol. We first complete the square using the terms in y and y^2 an write the given equation in the form $(y-k)^2 = 4a(x-h)$ wher (h,k) is the vertex and the focus is at $(h+a,k)$, the axis o symmetry is given by $y = k$ and the directrix is given by $x = h - a$

$2(y^2 + 4y) + x + 1 = 0$

$2((y+2)^2 - 4) + x + 1 = 0$

$(y+2)^2 = -(1/2)(x-7)$

vertex at $(7,-2)$

$-(1/2) = 4a$ hence, $a = -1/8$

focus at $(7-1/8,-2) = (6.875,-2)$

axis of symmetry is given by $y = -2$

directrix y =2 is equal to 2|a|, hence |a| =9 and a =-9 since parabola opens downward. The vertex is at (0,-7), at equal distances from the focus and the directrix.

equation: $x^2 = -36(y+7)$

Q11. Given the following equation

$$9x^2 + 4y^2 = 36$$

a) Find the x and y intercepts of the graph of the equation.

b) Find the coordinates of the foci.

c) Find the length of the major and minor axes.

Sol. a) We first write the given equation in standard form by dividing both sides of the equation by 36

$$9x^2 / 36 + 4y^2 / 36 = 1$$

$$x^2 / 4 + y^2 / 9 = 1$$

$$x^2 / 2^2 + y^2 / 3^2 = 1$$

We now identify the equation obtained with one of the standard equation in the review above and we can say that the given equation is that of an ellipse with **a=3** and **b=2** (NOTE: a>b)

Set y =0 in the equation obtained and find the x intercepts

$$x^2 / 2^2 = 1$$

Solve for x.

$$x^2 = 2^2$$

$$x = \pm 2$$

Set x =0 in the equation obtained and find the y intercepts.

$$y^2 / 3^2 = 1$$

Solve for y.

$$y^2 = 3^2$$

$$y = \pm 3$$

b) We need to find c first.

$$c^2 = a^2 - b^2$$

a and b were found in part a)

$$c^2 = 3^2 - 2^2$$

$$c^2 = 5$$

Solve for c.

$$c = \pm(5)^{1/2}$$

The foci are $F_1\left(0,(5)^{1/2}\right)$ and $F_2\left(0,-(5)^{1/2}\right)$

c) The major axis length is given by $2a = 6$

The minor axis length is given by $2b = 4$.

Q12. Find the equation to the locus of a moving point which is always equidistant from the points (2,-1) and (3,2). What curve does the locus represent?

Sol. Let A(2,-1) and B(3,2) be the given points and (x,y) be the co-ordinates of a point P on the required locus. Then,

$PA^2 = (x-2)^2 + (y+1)^2$ and $PB^2 = (x-3)^2 + (y-2)^2$

By problem, $PA = PB$ or $PA^2 = PB^2$

or, $(x-2)^2 + (y+1)^2 = (x-3)^2 + (y-2)^2$

or, $x^2 - 4x + 4 + y^2 + 2y + 1 = x^2 - 6x + 9 + y^2 - 4y + 4$

or, $2x + 6y = 8$

or, $x + 3y = 4$ ...... (1)

which is the required equation to the locus of the moving point. Clearly, equation (1) is a first degree equation in x and y; hence, the locus of P is a straight line whose equation is $x + 3y = 4$.

Q13. Find the locus of a moving point which forms a triangle of area 21 square units with the points (2,-7) and (-4,3).

Sol. Let the given point be A(2,-7) and B(-4,3) and the moving point

P (say), which forms a triangle of area 21 square units with A and B, have coordinates (x,y). Thus, by question area of the triangle PAB is 21 square units.

Hence, we have,

Therefore, the required equation to the locus of the moving point is $5x+3y=10$ or, $5x+3y+21=0$.

$$1/2\left|(6-4y-7x)-(28+3x+2y)\right|=21$$

or, $|6-28-4y-2y-7x-3x|=42$

or, $10x+6y+22=\pm 42$

Therefore, either, $10x+6y+22=42$ i.e., $5x+3y=10$

or, $10x+6y+22=-42$ i.e., $5x+3y+32=0$

Q14. Find the centre and radius of the following circle.

Sol. Let (h,k) be the centre of a circle with radius a.

Thus, its equation will be $(x-h)^2+(y-k)^2=a^2$

(i) Given:

$(x-1)^2+y^2=4$

Here, $h=1$, $k=0$ and $a=2$.

Thus, the centre is (1,0) and the radius is 2.

(ii) Given:

$(x+5)^2+(y+1)^2=9$

Here, $h=-5$, $k=-1$ and radius=3.

Thus, the centre is (-5,-1) and the radius is 3.

(iii) Given:

$x^2+y^2-4x+6y=5$

The given equation can be rewritten as follows:

$(x-2)^2+(y+3)^2-4-9=5$

UNIT-V

Statistic and Probability

Very Short Questions (On Law of Addition)

Q1. In a group of 101 students 30 are freshmen and 41 are sophomores. Find the probability that a student picked from this group at random is either a freshman or sophomore.

Sol. Note that P(freshman) = 30/101 and P(sophomore) = 41/101.

Thus P(freshman or sophomore) = 30/101 + 41/101 = 71/101.

This makes sense since 71 of the 101 students are freshmen or sophomores.

Q2. In a group of 101 students 40 are juniors, 50 are female and 22 are female juniors. Find the probability that a student picked from this group at random is either a junior or female.

Sol. Note that P(junior) = 40/101 and P(female) = 50/101 and P(junior and female) = 22/101. Thus

P(junior or female) = 40/101 + 50/101 -22/101 = 68/101

This makes sense since 68 of the 101 students are juniors or female.

Q3. An urn contains 6 red marbles and 4 black marbles. Two marbles are drawn *without replacement* from the urn. What is the probability that both of the marbles are black?

Sol. Let A = the event that the first marble is black; and let B= the event that the second marble is black. We know the following:

In the begining, there are 10 marbles in the urn, 4 of which are black. Therefore,

P(A) = 4/10

After the first selection, there are 9 marbles in the urn, 3 of which are black.

Therefore, P(B|A) = 3/9

Therefore, based on the rule of multiplication:

$$P(A \cap B) = P(A)P(B \mid A)$$

$$P(A \cap B) = (4/10)*(3/9) = 12/90 = 2/15$$

Q4. A card is drawn randomly from a deck of ordinary playing cards. You win \$10 if the card is a spade or an ace. What is the probability that you will win the game?

Sol. The correct answer is C. Let S = the event that the card is a spade, and let A = the event that the card is an ace. We know the following:

- There are 52 cards in the deck
- There are 13 spades, so P(S) = 13/52
- There are 4 aces, so P(A) = 4/52
- There is 1 ace that is also a spade, so $P(S \cap A) = 1/52$

Therefore, based on the rule of addition:

$$P(S \cup A) = P(S) + P(A) - P(S \cap A)$$

$$P(S \cup A) = 13/52 + 4/52 - 1/52 = 16/52 = 4/13$$

Q5. A card is drawn from a well shuffled pack of playing cards. Find the probability that

1. A club

2. A king

3. The ace of spade

Sol. There are 52 different cards in a pack of playing cards. Total number of cases $= 52C_1 = 52$

1. There will be 13 club cards in a packet. Therefore favourable cases$= 13C_1 = 13$. Hence probability of getting a club will be

$$\frac{13C_1}{52C_1} = \frac{13}{52}$$

2. There are 4 kings in a packet. Therefore number of favourable

cases= $4C_1$ and probability of getting a king will be

$$\frac{4C_1}{52C_1} = \frac{4}{52}$$

3. In a pack of cards there will be only one ace of spade and hence probability of getting an ace of spade will be

$$\frac{1}{52}$$

Q6. Three coins are tossed. What is the probability of getting

1. All heads

2. Exactly one head

3. Exactly two heads

4. Atleast one head

Sol. Here the sample space is S = (HHH, HHT, HTH, THH, TTH, THT, HTT, TTT) Therefore total number of cases is 8.

1. There is only one case with all heads as (HHH), therefore, number of favourable cases is 1.

Hence P(All heads) $= \frac{1}{8}$

2. There are three cases with exactly one head as (HTT, THT, TTH)

Hence P(Exactly one head) = $\frac{3}{8}$

3. There are three cases with exactly two heads as (HHT, HTH, THH)

Hence P(Exactly two heads) = $\frac{3}{8}$

4. In all 7 cases except (TTT) there is atleast one head.

Hence P(Atleast one head) = $\frac{7}{8}$

Very Short Question (on integration)

Q1. Calculate the mean for the following data
(i) 2,4,5,6,8,17

Sol. Mean = $\overline{X} = \frac{2+4+5+6+8+17}{6} = \frac{42}{6} = 7$

(ii) 6,7,10,12,13,4,8,12

Mean $= \frac{6+7+10+12+13+4+8+12}{8}$

$= \frac{72}{8}$

$= 9$

Q2. The weekly salaries of six employees at McDonalds are \$140, \$220, \$90, \$180, \$140, \$200. For these six salaries, find (a) the mean (b) the median (c) and the mode.

Sol. List the data in order: 90, 140, 140, 180, 200, 220

Mean $= \frac{90+140+140+180+200+220}{6} = 161.\overline{66}$

Median: 90, 140, 140, 180, 200, 220

The two numbers that fall in the middle need to be averaged.

$\frac{140+180}{2} = \mathbf{160}$

Mode: The number that appears the most is **140.**

Q3. Raj has grades of 84, 65 and 76 on three math tests. What grade must he obtain on the next test to have an average of exaclty 80 for the four tests?

Sol. $\frac{84+65+76+x}{4} = 80$ (average) cross multiply and solve

$(4)(80) = 225 + x$

$320 = 225 + x$

$\underline{-225} \quad \underline{-225}$

$95 = x$

Raj needs a 95 on his next test.

Q4. Calculate the mean deviation for the following
46, 44,54,63,34,38,70,55,48,42

Sol. NOTE:

Formula used for mean deviation:

$$MD = \frac{1}{n}\sum_{i=1}^{n} |d_i|$$

Here, $d_i = x_i - M$

M = Median

Arranging the data in ascending order

34, 38, 42, 44, 46, 48, 54, 55, 63, 70

Here, n =10

Also, median is the AM of the fifth and the sixth observation.

Median, $M = \frac{42+44}{2} = 43$

x_i	$\|d_i\| = \|x_i - M\|$
34	9
66	23
30	13
38	5
44	1
50	7
40	3
60	17
42	1
51	8
Total	87

$MD = \frac{1}{10} \times 87 = 8.7$

Q5. Calculate the mean deviation for the following
25, 29, 41, 28, 22, 31, 30, 24, 42, 27

Sol. Arranging the data in ascending order

22, 24, 25, 27, 28, 29, 30, 31, 41, 42

Here, n =10

Also, median is the AM of the fifth and the sixth observation.

Median, $M = \frac{28+29}{2} = 28.5$

x_i	$\lvert d_i \rvert = \lvert x_i - M \rvert$
22	6.5
24	4.5
30	1.5
27	1.5
29	0.5
31	2.5
25	3.5
28	0.5
41	12.5
41	13.5
Total	47

$MD = \frac{1}{10} \times 47 = 4.7$

Q6. The height of 50 plants in a garden are given below.

Height (cm)	10	25	30	40	45
Number Plants	13	15	12	8	2

Find the mode of the data

Sol. The frequency of 25 is maximum.

So, the mode of this data is 15.

Q7. If the mean of the following frequncy distributions is 9, find the value of 'a'. Write the tally marks also.

Variable (x_i)	4	6	8	10	12	15
Frequency (f_i)	8	9	17	a	8	4

Sol. Frequency distribution table

Variable (x_i)	Tally Marks	Frequency (f_i)	$f_i x_i$
4	卌 |||	8	32
6	卌 ||||	9	54
8	卌 卌 卌 ||	17	136
10		a	$10a$
12	卌 |||	8	96
15	||||	4	60
		$\sum f_i = 46 + a$	$\sum f_i x_i = 378 + 10a$

Mean = $\left(\sum f_i x_i\right) / \left(\sum f_i\right)$

But given mean = 9

So, we have (378 + 10a)/(46 + a) =9

378 + 10a = 9(46 + a)

378 + 10a = 414 + 9a

10a - 9a = 414 - 378

a = 36

Q8. Find the median of the data 24, 33, 30, 22, 21, 25, 34, 27.

Sol. Here the number of observations is even i.e., 8

Arranging the data in ascending order, we get 21, 22, 24, 25, 27, 30, 33, 34

Therefore, median = {$(n/2)^{th}$ observation + $(n + 1/2)^{th}$ observation}/2

= $(8/2)^{th}$ observation + $(8/2+1)^{th}$ observation/2

= 4^{th} observation + $(4+1)^{th}$ observation

={25+27}/2

=52/2

=26

Therefore, the median of the given data is 26

Q9. Find the median of the data 25, 37, 47, 18, 19, 26, 36

Sol. Arranging the data in ascending order, we get 18, 19, 25, 26, 36, 37, 47

Here the number of observations is odd, i.e., 7

Therefore, median = $(n + 1/2)^{th}$ observation

= $(7 + 1/2)^{th}$ observation

= $(8/2)^{th}$ observation

= 4^{th} observation

4^{th} observation is 26

Therefore, the median of the data is 26.

Q10. The height of 10 girls were measured in cm and results are as follows:

142, 149, 136, 148, 129, 140, 148, 145, 150, 133

(i) What is the height of tallest girl?

(ii) What is the height of shortest girl?

(iii) What is the range of the data?

(iv) Find the mean height?

(v) How many girls are there whose height is less than the mean height?

Sol. (i) The height of the tallest girl is 150 cm.

(ii) The height of the shortest girl is 129 cm.

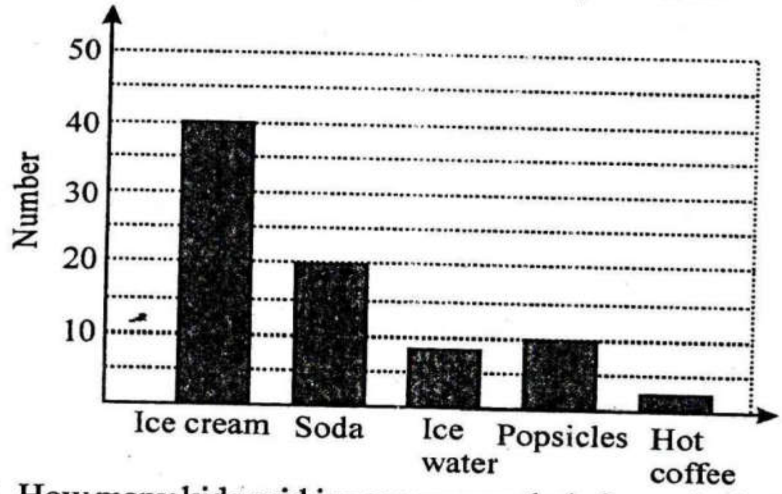
(iii) Range = 150cm - 129cm = 21 cm

(iv) The mean height = (142+149+136+148+129+140+148+145+150+133)/10 = 1420/10 = 142 cm

(v) There are 4 girls whose height is less than the mean height, i.e., the girl having heights 136 cm, 129 cm, 133 cm, 140 cm.

Q11. A survey was conducted of 79 kids at a local swimming pool to find their favourite hot-weather refreshment. Use

the bar graph below to answer the questions.



1. How many kids said ice cream was their favourite hot weather refreshment?

2. Which refreshment was the favourite of exactly 20 kids?

3. Which was more popular, popsicles or ice water?

4. How many kids said hot coffee was their favourite hot weather refreshment?

Sol. 1. 40, since that's the height of the bar in the "ice cream" bin.

2. Soda, since there are 20 kids in the "soda" bin.

3. Popsicles, since the bar above "popsicles" is slightly taller than the bar above "ice water"

4. Only one

Short Question

Q1. If the mean of the following distribution is 9, find the value of p.

x	4	6	$p+7$	10	15
f	5	10	10	7	8

Sol. Calculation of mean

x_i	f_i	$x_i f_i$
4	5	20
6	10	60
$p+7$	10	$10(p+7)$
10	7	70
15	8	120

$\sum f_i = 5+10+10+7+8 = 40$

$\sum f_i x_i = 270 + 10(p+7)$

Mean$= \sum(f_i x_i) / \sum f_i$

$9 = \{270 + 10(p+7)\} / 40$

$\Rightarrow 270 + 10p + 70 = 9 \times 40$

$\Rightarrow 340 + 10p = 360$

$\Rightarrow 10p = 360 - 340$

$\Rightarrow 10p = 20$

$\Rightarrow p = 20/10$

$\Rightarrow p = 2$

Q2. In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that vowels must always come together?

Sol. The word 'MATHEMATICS' has 11 letters. It has the vowels 'A', 'E', 'A', 'I' in it and these 4 vowels must always come together. Hence these 4 vowels can be grouped and considered as a single letter. That is, MTHMTCS(AEAI)

Hence, we can assume total letters as 8. But in these 8 letters, 'M' occurs as 2 times, 'T' occurs 2 times but rest of the letters are different.

Hence, number of ways to arrange these letters

$=8!(2!)(2!)=8\times7\times6\times5\times4\times3\times2\times1(2\times1)(2\times1)=10080$

In the 4 vowels (AEAI), 'A' occurs 2 times and rest of the vowels are different.

Number of ways to arrange these vowels among themselves

$=4!2!=4\times3\times2\times1(2\times1)=12$

Hence, required number of ways $=10080\times12=120960$

Q3. A question paper has two parts P and Q, each containing 10 questions. If a student needs to choose 8 from part P and 4 from part Q, in how many ways can he do that?

Sol. Number of ways to choose 8 questions from part $P={}^{10}C_8$

Number of ways to choose 4 questions from part $Q={}^{10}C_4$

Total number of ways

$={}^{10}C_8\times{}^{10}C_4$

$={}^{10}C_2\times{}^{10}C_4$ [Applied the formula ${}^nC_r={}^nC_{(n-r)}$]

$=((10\times9)/(2\times1))((10\times9\times8\times7)/(4\times3\times2\times1))$

$=45\times210$

$=9450$

Q4. A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

Sol. Total number of balls = (2+3+2) = 7

Let S be the sample space.

Then, n(S) = Number of ways of drawing 2 balls out of 7

$={}^7C_2$

$=\frac{(7\times6)}{(2\times1)}$

$=21$

Let E = Event of drawing 2 balls, none of which is blue.

$\therefore n(E) =$ Number of ways of drawing 2 balls out of (2+3) balls.

$= {}^5C_2$

$= \frac{(5\times4)}{(2\times1)} = 10 \therefore P(E) = \frac{n(E)}{n(S)} = \frac{10}{21}$

Q5. Find the mean of following frequency distribution by step deviation method.

Class interval	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
Number of workers	7	10	15	8	10

Sol.

Class interval	Mid values (x_i)	Frequency (f_i)	$d_i = x_i - 25$	$u_i = (x_i - 25)/10$	$f_i u_i$
0 – 10	5	7	–20	–2	–14
10 – 20	15	10	–10	–1	–10
20 – 30	25	15	0	0	0
30 – 40	35	8	10	1	8
40 – 50	45	10	20	2	20
		$N = \sum f_i = 50$			$\sum f_i u_i = 4$

A = 25, h = 10, N = 50 and $\sum f_i u_i = 4$

$\overline{X} = A + h\left[\frac{1}{N}\sum f_i u_i\right]$

$\Rightarrow$ Mean = 25+10×(4 / 50)

$\Rightarrow$ Mean = 25+0.8

$\therefore$ Mean = 25.8

Q6. The following table shows the weights of 12 students:

Weight (in kg)	67	70	72	73	75
Number of students	4	3	2	2	1

Find the mean by using short-cut method.

Sol. Let the assumed mean = A = 72

Weight (in kg)	No of students (f_i)	$d_i = x_i - A = x_i - 72$	$f_i d_i$
67	4	−5	−20
70	3	−2	−6
72	2	0	2
73	2	1	2
75	1	3	3
	$\sum f_i = 12$		$\sum f_i d_i = -21$

$\sum f_i = 12, \sum f_i d_i = -21, A = 72$

$$\overline{X} = A + \frac{1}{N}\sum f_i u_i$$

$\Rightarrow$ Mean $= 72 + (-21)/12 = 72 - 7/4$

$\Rightarrow$ Mean $= 70.25kg$

Q7. Calculate the variance, standard deviation from the following data:

x	3	8	13	18	23
y	7	10	15	10	6

Sol.

x_i	f_i	$f_i x_i$	$(x_i - \overline{X})$	$(x_i - \overline{X})^2$	$f_i(x_i - \overline{X})^2$
3	7	21	−9.79	95.84	670.88
8	10	80	−4.79	22.94	229.4
13	15	195	0.21	0.04	0.6
18	10	180	5.21	27.14	271.4
23	6	138	10.21	104.24	625.44
	$\sum f_i = 48$	$\sum f_i x_i = 614$			$\sum f_i(x_i - \overline{X})^2 = 1797.32$

$$\text{Variance, } \sigma^2 = \frac{\sum f_i (x_i - \overline{X})^2}{\sum f_i} = \frac{1797.32}{48} = 37.44$$

$SD, \sigma = \sqrt{37.44} = 6.12$

Formulas

Calculate?	Formula Name	Formula
Mean	Direct Method	$\bar{x} = \frac{\sum x}{n}$
Mean	Step Deviation Method	$\bar{x} = A + h\left[\frac{1}{N}\sum f_i u_i\right]$
Mean	Short Cut Method	$\bar{x} = A + \frac{1}{N}\sum f_i u_i$
Variance	σ^2	$\frac{\sum f_i (x_i - \overline{X})^2}{\sum f_i}$
Standard Deviation	SD	$\sqrt{Var(X)}$

Q8. Compute the following

$$\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

Sol. $$\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x + \sin^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \cos^2 x + \sin^2 x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$(\because \sin^2 x + \cos^2 x = 1)$

Q9. C. .npute the indicated products

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Sol. (i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} a(a)+b(b) & a(-b)+b(a) \\ -b(a)+a(b) & -b(-b)+a(a) \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2 & -ab+ab \\ -ab+ab & b^2+a^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$$

Q10. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Sol. (i) Let $A = \begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ then $A^T = \begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$

(ii) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

Long Questions

Q1. The table below gives the percentage distribution of female teachers in the primary schools of rural areas of various states and union territories (U.T.) of India. Find the mean percentage of female teachers by all the three methods.

Direct method

Assumed

Step- Deviation

Percentage of female teachers	15 – 25	25 – 35	35 – 45	45 – 55	55 – 65	65 – 75	75 – 85
Number of States/U.T.	6	11	7	4	4	2	1

Sol. Let us find the class marks, x_i of each class, and put them in a column (see Table)

Table

Percentage of female teachers	Number of States/U.T.(f_i)	x_i
15 – 25	6	20
25 – 35	11	30
35 – 45	7	40
45 – 55	4	50
55 – 65	4	60
65 – 75	2	70
75 – 85	1	80

Here we take a = 50, h=10 then $d_i = x_i - 50$ and $u_i = \frac{x_i - 50}{10}$

Percentage of female teachers	Number of States/U.T.(f_i)	x_i	$d_i = x_i - 50$	$u_i = \frac{x_i - 50}{10}$	$f_i x_i$	$f_i d_i$	$f_i u_i$
15 – 25	6	20	–30	–3	120	–180	–18
25 – 35	11	30	–20	–2	330	–220	–22
35 – 45	7	40	–10	–1	280	–70	–7
45 – 55	4	50	0	0	200	0	0
55 – 65	4	60	10	1	240	40	4
65 – 75	2	70	20	2	140	40	4
75 – 85	1	80	30	3	80	30	3
Total	35				1390	–360	–36

From the table above, we obtain $\sum f_i = 35, \sum f_i x_i = 1390,$

$\sum f_i d_i = -360, \sum f_i u_i = -36$

Using the direct method, $\bar{x} = \frac{\sum f_i x_i}{\sum f_i} = \frac{1390}{35} = 39.71$

Using the assumed mean method,

$$\bar{x} = a + \frac{\sum f_i d_i}{\sum f_i} = 50 + \frac{(-360)}{35} = 39.71$$

Using the step-deviation method,

$$\bar{x} = a + \left(\frac{\sum f_i u_i}{\sum f_i}\right) \times h = 50 + \frac{(-36)}{35} \times 10 = 39.71$$

Q2. Find the mean of the following data by, assumed mean method

Class interval	10-25	25-40	40-55	55-70	70-85	85-100
No. of student	2	3	7	6	6	6

Sol. : Let the assumed mean is (a) = 47.5

Class interval	Number of Student (d_i)	Class Mark (x_i)	Deviation $d_i = x_i - a$	$f_i d_i$
10-25	2	17.5	-30	-60
25-40	3	32.5	-15	-45
40-55	7	47.5	0	0
55-70	6	62.5	15	90
70-85	6	77.5	30	180
85-100	6	92.5	45	270
Total	$\Sigma f_i = 30$			$\Sigma f_i d_i = 435$

Mean $\overline{x} = a + \frac{\Sigma f_i d_i}{\Sigma f_i} = 47.5 + \frac{435}{30} = 47.5 + 14.5 = 62$ (Ans.)

Shortcut Method:-

$\overline{X} = a + \frac{\sum fu}{\sum f} \times h$ where $u = \frac{d}{n}$

where $\overline{x}$ = mean
h = class interval
a = assumed mean

$u = \frac{d}{h}$

d= deviation = $\overline{x}$ - a

$\sum fd$ = sum of multiplication of deviation and corresponding frequency.

$\sum f$ = Sum of frequencies.

Q3. The following observations have been arranged in ascending order. If the median of the data is 63, find the value of *x*.
29, 32, 48, 50, x, x+2, 72, 78, 84, 95,

Sol. Here, N = 10

$$\text{Median} = \frac{1}{2} \times \left[\left(\frac{N}{2}\right)^{th} + \left(\frac{N}{2}+1\right)^{th}\right] obs$$

$$63 = \frac{1}{2} \times \left[\left(\frac{10}{2}\right)^{th} + \left(\frac{10}{2}+1\right)^{th}\right] obs$$

$$\Rightarrow 63 = \frac{1}{2} \times \left[5^{th} + 6^{th}\right] obs$$

$$\Rightarrow 63 = \frac{1}{2} \times \left[x + x + 2\right]$$

$$\Rightarrow \frac{2x+2}{2} = 63 \Rightarrow x+1 = 63 \qquad \Rightarrow x = 62$$

Q4. **The number of telephone calls received at an exchange per interval for 24 successive one minute interval are shown in the following table :**

Number of calls	**Frequency**
0	**14**
1	**21**
2	**25**
3	**43**
4	**51**
5	**40**
6	**39**
7	**12**

Calculate mean, median and mode. **(Raj BCA 2008)**

Sol. :

Number of calls (x)	Frequency (f)	fx	$c.f.$
0	14	0	14
1	21	21	35
2	25	50	60
3	43	129	103
4	51	204	154
5	40	200	194
6	39	234	233
7	12	84	245
	$\Sigma f = 245$	$\Sigma fa = 922$	

Mean: $\bar{X} = \frac{\sum fx}{\sum f}$

$$= \frac{922}{145} = 3.763$$

Median : For finding the median first we calculate $\frac{N+1}{2}$ (where $N = \Sigma f$)

i.e. $\frac{245+1}{2} = \frac{246}{2}$

Now, c.f just greater than 123 is 154 and the corresponding variate to 154 is 4 so median is 4.

Mode : The maximum frequency is 51 of variate 4 so mode is 4.

Q5. Find the mode for the following frequency distribution of marks obtained by 80 students

Marks	No. of Students
0-10	6
10-20	10
20-30	12
30-40	32
40-50	20

← **Model Class** (30-40)

Sol. Since, class 30-40 has the maximum frequency so, this is the modal class.

Then, $f_1 = 32$, $f_o = 12$, $f_2 = 20$, $l=30$, $h=10$

$$\text{Mode} = l + \left(\frac{f_1 - f_o}{2f_1 - f_o - f_2}\right) \times h$$

$$= 30 + \left(\frac{32-12}{2\times 32-12-20}\right)\times 10$$

$$= 30 + \left(\frac{20}{64-32}\right)\times 10$$

$$= 30 + \frac{200}{32} = 30 + \frac{25}{4}$$

$$= 30 + 6.25 = 36.25 \quad \text{(Ans.)}$$

Q6. The points scored by a Kabaddi team in a series of matches are as follows:

17, 2, 7, 27, 15, 5, 14, 8, 10, 24, 48, 10, 8, 7, 18, 28,

Find the median of the points scored by the team

Sol : Arranging the points scored by the team in ascending order, we get.

2, 5, 7, 7, 8, 8, 10, 10, 14, 15, 17, 18, 24, 27, 28, 48,

Here, N = 16

$$\therefore \text{Median} = \frac{1}{2}\times\left[\left(\frac{N}{2}\right)^{th} + \left(\frac{N}{2}+1\right)^{th}\right] obs$$

$$= \frac{1}{2}\times\left[\left(\frac{16}{2}\right)^{th} + \left(\frac{16}{2}+1\right)^{th}\right] obs$$

$$= \frac{1}{2}\times\left[8^{th} + 9^{th}\right] \text{ obs} = \frac{1}{2}\times[10+14] = 12$$

Median of Ungrouped Data (If frequency is given):

» Find c.f. of given data.

» Find $\frac{N+1}{2}$.

» Determine c.f. just greater than $\frac{N+1}{2}$ corresponding variate of this c.f will be median.

Q7. The heights (in cm) of a students of a class are as follows:-

155	160	145	149	150	147	152	144	148

Find the median of this data.

Sol: On arranging the data in ascending order
144, 145, 147, 148, 149, 150, 152, 155, 160
Here, N = 9

$$\therefore \text{Median} = \left(\frac{N+1}{2}\right)^{th} obs = \left(\frac{9+1}{2}\right)^{th} obs. = 5^{th}\ obs. = 149 \text{ cm.}$$

Q7. $^{10}p_r$ **= 5040 find the value of *r*.**

Sol : $^{10}p_r = 5040$

$$\Rightarrow \frac{10!}{(10-r)!} = 10\times9\times8\times7 \qquad \left(\text{using } {}^{n}p_r = \frac{n!}{(n-r)!}\right)$$

$$\Rightarrow \frac{10!}{(10\text{-}r)!} = \frac{10\times9\times8\times7\times6!}{6!}$$

$$\Rightarrow \frac{10!}{(10-r)!} = \frac{10!}{6!}$$

$\Rightarrow (10 - r)! = 6!$
$\Rightarrow 10 - r = 6$
$\Rightarrow r = 10 - 6 = 4$

Q8. A box contain 6 red, 4 white and 5 black balls. 4 balls are drawn from the box at random. Find the probability that among the balls drawn there is at least one ball of each colous.

Sol.: Total number of balls = 6 + 4 + 5 = 15

4 ball scan bedra/vn = $^{15}c_4$ wags

We know that there is at least one ball of each colour. It may be passible in following cases :

2 Red, 1 white and 1 black:

It may passible in $^6c_2$ x ^4c (x 5Cj ways (ii) 1 Red. 2 white and 1 black:

It may be possible in 6c, x 4c2 x 5C| ways.)

1 Red, 1 white and 2 black:

It may be possible in

$^6c_1 \times {}^4c_1 \times {}^5c_2$

So probability

$$=\frac{\left({}^6C_2 \times {}^4C_1 \times {}^5C_1 + {}^6C_1 \times {}^4C_2 \times {}^5C_1 + {}^6C_1 \times {}^4C_1 \times {}^5C_2\right)}{{}^{15}C_4}$$

$$=\frac{15\times4\times5\times6\times6\times4\times10}{1364}$$

$=0.527$

Q9. A bag contains six black and four red balls. Two balls are drawn. Find the probability that at least one is red.

Solution : Total number of balls 6 + 4= 10.

There are two possibilities for getting at least one is red.

Both are red (ii) one is red and one is black

So, required probability = P(both red) + P(one red and one black)

$$=\frac{{}^4C_2}{{}^{10}C_2}+\frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2}$$

$$= \frac{12 + 48}{90} = \frac{60}{90} = \frac{2}{3}$$

Q10. If $X = 4Y + 5$ and $Y = kX + 4$ are two lines of regression of X on Y and Y on X respectively: Show that $0 < 4k < 1$ and If $k = 1/16$, find the correlation coefficient r.

Sol: (i) Given Regression line

X on Y $\rightarrow X = 4Y + 5$

so $b_{xy} = 4$

and Y on X $\rightarrow Y = KX + 4$

so $b_{yx} = K$

we know that

$0 < b_{xy} \times b_{yx} < 1$

$\rightarrow 0 < 4K < 1$ ....(1) (HP)

(ii) $K = \frac{1}{16}$

Now, $r = \pm\sqrt{b_{xy} \times b_{yx}}$

$r = \pm\sqrt{4K}$

$K = \frac{1}{16}$

$r = \sqrt{4 \times \frac{1}{16}}$

$= \sqrt{\frac{1}{4}}$

$= \frac{1}{2}$