Biyani's Think Tank

Concept based notes

Mathematical Methods for Numerical Analysis and Optimization

(MCA)

Varsha Gupta
Poonam Fatehpuria
M.Sc. (Maths)
Lecturer
Deptt. of Information Technology
Biyani Girls College, Jaipur

Biyani's Group of Girls' Colleges

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Preface

I am glad to present this book, especially designed to serve the needs of the students. The book has been written keeping in mind the general weakness in understanding the fundamental concepts of the topics. The book is self-explanatory and adopts the “Teach Yourself” style. It is based on question-answer pattern. The language of book is quite easy and understandable based on scientific approach.

This book covers basic concepts related to the microbial understandings about diversity, structure, economic aspects, bacterial and viral reproduction etc.

Any further improvement in the contents of the book by making corrections, omission and inclusion is keen to be achieved based on suggestions from the readers for which the author shall be obliged.

I acknowledge special thanks to Mr. Rajeev Biyani, *Chairman* & Dr. Sanjay Biyani, *Director* (*Acad.*) Biyani Group of Colleges, who are the backbones and main concept provider and also have been constant source of motivation throughout this Endeavour. They played an active role in coordinating the various stages of this Endeavour and spearheaded the publishing work.

I look forward to receiving valuable suggestions from professors of various educational institutions, other faculty members and students for improvement of the quality of the book. The reader may feel free to send in their comments and suggestions to the under mentioned address.

Author

Syllabus

B.C.A. Part-II

Mathematical Methods for Numerical Analysis and Optimization

Computer arithmetics and errors. Algorithms and programming for numerical solutions. The impact of parallel computer : introduction to parallel architectures. Basic algorithms Iterative solutions of nonlinear equations : bisection method, Newton-Raphson method, the Secant method, the method of successive approximation. Solutions of simultaneous algebraic equations, the Gauss elimination method. Gauss-Seidel Method, Polynomial interpolation and other interpolation functions, spline interpolation system of linear equations, partial pivoting, matrix factorization methods. Numerical calculus : numerical differentiating, interpolatory quadrature. Gaussian integration. Numerical solutions of differential equations. Euler's method. Runge-Kutta method. Multistep method. Boundary value problems : shooting method.

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Content

S.No.	Name of Topic
16.	Newton's Formula for Backward Interpolation
17.	Divided Difference Interpolation
18.	Lagrange's Interpolation
19.	Spline Interpolation
20.	Quadratic Splines
21.	Cubic Splines
22.	Numerical Differentiation
23.	Numerical Integration
24.	Euler's Method
25.	Euler's Modified Method
26.	Rungs – Kutta Method
27.	Shooting Method

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Chapter-1

Computer Arithmetic and Errors

Q.1. An approximate value of π is given by x_1 = 22/7 = 3.1428571 and its true value is x = 3.1415926. Find the absolute and relative errors.

Ans.: True value of π(x) = 3.1415926

Approximate value of π(x_1) = 3.1428571

Absolute error is given by –

$$E_a = |x - x_1|$$

$$= |3.1415926 - 3.1428571|$$

$$= 0.0012645$$

Relative error is given by –

$$E_r = \left|\frac{x - x_1}{x}\right|$$

$$= \left|\frac{3.1415926 - 3.1428571}{3.1415926}\right|$$

$$= \left|\frac{0.0012645}{3.1415926}\right|$$

$$= 0.0004025$$

Q.2. Let x = 0.00458529 find the absolute error if x is truncated to three decimal digits.

Ans.: $x = 0.00458529 = 0.458529 \times 10^{-2}$ [in normalized floating point form]

$x_1 = 0.458 \times 10^{-2}$ [after truncating to three decimal places]

Absolute error $= |x - x_1|$

$= |0.458529 \times 10^{-2} - 0.458 \times 10^{-2}|$

$= 0.000529 \times 10^{-2}$

$= 0.000529 \text{ E} - 2$

$= 0.529 \text{ E} - 5$

Q.3. Let the solution of a problem be x_a = 35.25 with relative error in the solution atmost 2% find the range of values upto 4 decimal digits, within which the exact value of the solution must lie.

Ans.: We are given that the approximate solution of the problem is $(x_a) = 35.25$ and it has relative error upto 2% so

$$\left|\frac{x - 35.25}{x}\right| < 0.02$$

$$= -0.02 < \frac{x - 35.25}{x} < 0.02$$

Case-I : if $-0.02x < \frac{x - 35.25}{x}$

$\Rightarrow -0.02x < x - 35.25$

$\Rightarrow 35.25 < x + 0.02x$

$\Rightarrow 35.25 < x(1 + 0.02)$

$\Rightarrow 35.25 < x(1.02)$

$\Rightarrow 35.25 < 1.02x$

$\Rightarrow \frac{35.25}{1.02} < x$

$\Rightarrow x > 34.5588$ _ _ _ (1)

Case-II: if $\frac{x - 35.25}{x} < 0.02$

$\Rightarrow x - 35.25 < 0.02x$

$\Rightarrow x - 0.02x < 35.25$

$\Rightarrow 0.98x < 35.25$

$\Rightarrow x < \frac{35.25}{0.98}$

$\Rightarrow x < 35.9693$ _ _ _ (2)

From equation (1) and (2) we have $34.5588 < x < 35.9693$

$\therefore$ The required range is (34.5588, 35.9693)

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Chapter-2

Bisection Method

Q.1. Find real root of the equation $x^3 - 5x + 3$ upto three decimal digits.

Ans.: Here $f(x) = x^3 - 5x + 3$

$f(0) = 0 - 0 + 3 = 3\ f(x_0)$ (say)

$f(1) = 1 - 5 + 3 = -1 = f(x_1)$ (say)

Since $f(x_0), f(x_1) < 0$ so the root of the given equation lies between 0 and 1

So, $x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{0+1}{2} = 0.5$

Now, $f(x_2) = f(0.5)$

$= (0.5)^3 - 5(0.5) + 3$

$= 0.125 - 2.5 + 3$

$= 0.625$ (which is positive)

$\because \quad f(x_1).f(x_2) < 0$

So, $x_3 = \dfrac{x_1 + x_2}{2} = \dfrac{1 + 0.5}{2} = 0.75$

Now, $f(x_3) = f(0.75)$

$= (0.75)^3 - 5(0.75) + 3$

$= 0.4218 - 3.75 + 3$

$= -0.328$ (which is negative)

$\because \quad f(x_2).f(x_3) < 0$

So, $x_4 = \frac{x_2 + x_3}{2} = \frac{0.5 + 0.75}{2} = 0.625$

Now, $f(x_4) = f(0.625)$

$= (0.625)^3 - 5\,(0.625) + 3$

$= 0.244 - 3.125 + 3$

$= 0.119$ (which is positive)

$\because \quad f(x_3).f(x_4) < 0$

So, $x_5 = \frac{x_3 + x_4}{2} = \frac{0.75 + 0.625}{2} = 0.687$

Now, $f(x_5) = f(0.687)$

$= (0.687)^3 - 5\,(0.687) + 3$

$= -\,0.1108$ (which is negative)

$\because \quad f(x_4).f(x_5) < 0$

So, $x_6 = \frac{x_4 + x_5}{2} = \frac{0.625 + 0.687}{2} = 0.656$

Now, $f(x_6) = f(0.656)$

$= (0.656)^3 - 5\,(0.656) + 3$

$= 0.0023$ (which is positive)

$\because \quad f(x_5).f(x_6) < 0$

So, $x_7 = \frac{x_5 + x_6}{2} = \frac{0.687 + 0.656}{2} = 0.671$

Now, $f(x_7) = f(0.671)$

$= (0.671)^3 - 5\,(0.671) + 3$

$= -\,0.0528$ (which is negative)

$\because \quad f(x_6).f(x_7) < 0$

So, $x_8 = \frac{x_6 + x_7}{2} = \frac{0.656 + 0.671}{2} = 0.663$

Now, $f(x_8) = f(0.663)$

$= (0.663)^3 - 5\,(0.663) + 3$

$= 0.2920 - 3.315 + 3$

$= -0.023$ (which is negative)

$\because \quad f(x_6).f(x_8) < 0$

So, $x_9 = \frac{x_6 + x_8}{2} = \frac{0.656 + 0.663}{2} = 0.659$

Now, $f(x_9) = f(0.659)$

$= (0.659)^3 - 5\,(0.659) + 3$

$= -0.0089$ (which is negative)

$\because \quad f(x_6).f(x_9) < 0$

So, $x_{10} = \frac{x_6 + x_9}{2} = \frac{0.656 + 0.659}{2} = 0.657$

Now, $f(x_{10}) = f(0.657)$

$= (0.657)^3 - 5\,(0.657) + 3$

$= -0.00140$ (which is negative)

$\because \quad f(x_6).f(x_{10}) < 0$

So, $x_{11} = \frac{x_6 + x_{10}}{2} = \frac{0.656 + 0.657}{2} = 0.656$

Now, $f(x_{11}) = f(0.656)$

$= (0.656)^3 - 5\,(0.656) + 3$

$= 0.2823 - 3.28 + 3$

$= 0.00230$ (which is positive)

$\because \quad f(x_{11}).f(x_{10}) < 0$

So, $x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{0.657 + 0.656}{2} = 0.656$

Since x_{11} and x_{12} both same value. Therefore if we continue this process we will get same value of x so the value of x is 0.565 which is required result.

Q.2. Find real root of the equation cos x - xe^x = 0 correct upto four decimal places.

Ans.: Since, $f(x) = \cos x - xe^x$

So, $f(0) = \cos 0 - 0e^0 = 1$ (which is positive)

And $f(1) = \cos 1 - 1e^1 = -2.1779$ (which is negative)

$\because$ $f(0).f(1) < 0$

Hence the root of are given equation lies between 0 and 1.

let $f(0) = f(x_0)$ and $f(1) = f(x_1)$

So, $x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{0 + 1}{2} = 0.5$

Now, $f(x_2) = f(0.5)$

$f(0.5) = \cos(0.5) - (0.5)e^{0.5}$

$= 0.05322$ (which is positive)

$\because$ $f(x_1).f(x_2) < 0$

So, $x_3 = \dfrac{x_1 + x_2}{2} = \dfrac{1 + 0.5}{2} = \dfrac{1.5}{2} = 0.75$

Now, $f(x_3) = f(0.75)$

$= \cos(0.75) - (0.75)e^{0.75}$

$= - 0.856$ (which is negative)

$\because$ $f(x_2).f(x_3) < 0$

So, $x_4 = \dfrac{x_2 + x_3}{2} = \dfrac{0.5 + 0.75}{2} = 0.625$

$f(x_4) = f(0.625)$

$= \cos(0.625) - (0.625)e^{(0.625)}$

$= - 0.356$ (which is negative)

$\because$ $f(x_2).f(x_4) < 0$

So, $x_5 = \dfrac{x_2 + x_4}{2} = \dfrac{0.5 + 0.625}{2} = 0.5625$

Now, $f(x_3) = f(0.5625)$

$= \cos(0.5625) - 0.5625e^{0.5625}$

$= -0.14129$ (which is negative)

$\because \quad f(x_2).f(x_5) < 0$

So, $x_6 = \dfrac{x_2 + x_5}{2} = \dfrac{0.5 + 0.5625}{2} = 0.5312$

Now, $f(x_6) = f(0.5312)$

$= \cos(0.5312) - (0.5312)e^{0.5312}$

$= -0.0415$ (which is negative)

$\because \quad f(x_2).f(x_6) < 0$

So, $x_7 = \dfrac{x_2 + x_6}{2} = \dfrac{0.5 + 0.5312}{2} = 0.5156$

Now, $f(x_7) = f(0.5156)$

$= \cos(0.5156) - (0.5156)e^{0.5156}$

$= 0.006551$ (which is positive)

$\because \quad f(x_6).f(x_7) < 0$

So, $x_8 = \dfrac{x_6 + x_7}{2} = \dfrac{0.513 + 0.515}{2} = 0.523$

Now, $f(x_8) = f(0.523)$

$= \cos(0.523) - (0.523)e^{0.523}$

$= -0.01724$ (which is negative)

$\because \quad f(x_7).f(x_8) < 0$

So, $(x_9) = \dfrac{x_7 + x_8}{2} = \dfrac{0.515 + 0.523}{2} = 0.519$

Now, $f(x_9) = f(0.519)$

$= \cos(0.519) - (0.519)e^{0.519}$

$= -0.00531$ (which is negative)

$\because \quad f(x_7).f(x_9) < 0$

So, $\quad (x_{10}) = \dfrac{x_7 + x_9}{2} = \dfrac{0.515 + 0.519}{2} = 0.5175$

Now, $f(x_{10}) = f(0.5175)$

$= \cos(0.5175) - (0.5175)e^{0.5175}$

$= 0.0006307$ (which is positive)

$\because \quad f(x_9).f(x_{10}) < 0$

So, $\quad x_{11} = \dfrac{x_9 + x_{10}}{2} = \dfrac{0.5195 + 0.5175}{2} = 0.5185$

Now, $f(x_{11}) = f(0.5185)$

$= \cos(0.5185) - (0.5185)e^{0.5185}$

$= -0.002260$ (which is negative)

$\because \quad f(x_{10}).f(x_{11}) < 0$

So, $\quad x_{12} = \dfrac{x_{10} + x_{11}}{2} = \dfrac{0.5175 + 0.5185}{2} = 0.5180$

Hence the root of the given equation upto 3 decimal places is x = 0.518

Thus the root of the given equation is x = 0.518

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Chapter-3

Regula Falsi Method

Q.1. Find the real root of the equation x $\log_{10}$ x – 1.2 = 0 correct upto four decimal places.

Ans.: Given $f(x) = x \log_{10} x - 1.2$ _ _ _ (1)

In this method following formula is used -

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})\ f(x_n)}{(f(x_n) - f(x_{n-1}))} \quad \_\ \_\ \_\ (2)$$

Taking x = 1 in eq.(1)

$f(1) = 1.\log_{10}1 - 1.2$

$= -2$ (which is negative)

Taking x = 2 in eq.(1)

$f(2) = 2.\log_{10}2 - 1.2$

$= -0.5979$ (which is negative)

Taking x = 3 in eq.(1)

$f(3) = 3.\log_{10}3 - 1.2$

$= 0.2313$ (which is positive)

$\because \quad f(2).f(3) < 0$

So the root of the given equation lies between 2 and 3.

let $x_1 = 2$ and $x_2 = 3$

$\therefore \quad f(x_1) = f(2) = -0.5979$

And $f(x_2) = f(3) = 0.2313$

Now we want to find x_3 so using eq.(2)

$$x_3 = x_2 - \frac{(x_2 - x_1)\ f(x_2)}{f(x_2) - f(x_1)}$$

$$= 3 - \frac{(3-2) \times (0.2313)}{0.2313 - (-0.5979)}$$

$$= 3 - \frac{0.2313}{0.8292}$$

$$= 3 - 0.2789 = 2.7211$$

$$f(x_3) = f(2.7211)$$

$$= 2.7211 \log_{10} 2.7211 - 1.2$$

$$= -0.01701 \text{ (which is negative)}$$

$\because \quad f(x_2).f(x_3) < 0$

Now to find x_4 using equation (2)

$$x_4 = x_3 - \frac{(x_3 - x_2)\ f(x_3)}{f(x_3) - f(x_2)}$$

$$= 2.7211 - \frac{(2.7211 - 3) \times (-0.0170)}{(-0.0170 - 0.2313)}$$

$$= 2.7211 - \frac{0.004743}{0.2483}$$

$$= 2.7211 + 0.01910 = 2.7402$$

Now

$$f(x_4) = f(2.7402)$$

$$= 2.7402 \log_{10} 2.7402 - 1.2$$

$$= -0.0003890 \text{ (which is negative)}$$

$\because \quad f(x_2).f(x_4) < 0$

Now to find x_5 using equation (2)

$$x_5 = x_4 - \frac{(x_4 - x_2)\ f(x_4)}{[f(x_4) - f(x_2)]}$$

$$= 2.7402 - \frac{(2.7402 - 3)}{(-0.0004762 - 0.2313)} \times (-0.0004762)$$

$$= 2.7402 + \frac{(-0.2598)(-0.0004762)}{0.2317}$$

$$= 2.7402 + \frac{(0.0001237)}{0.2317}$$

$$= 2.7402 + 0.0005341 = 2.7406$$

$f(x_5) = f(2.7406)$

$= 2.7406 \log_{10} 2.7406 - 1.2$

$= - 0.0000402$ (which is negative)

$\because \quad f(x_2).f(x_5) < 0$

To find x_6 using equation (2)

$$x_6 = x_5 - \frac{(x_5 - x_2)\ f(x_5)}{f(x_5) - f(x_2)}$$

$$= 2.7406 + \frac{(2.7406 - 3)\ \times\ (-0.000040)}{(-0.00004) - (0.2313)}$$

$$= 2.7406 + 0.000010 = 2.7406$$

$\therefore$ The approximate root of the given equation is 2.7406 which is correct upto four decimals.

Q.2. Find the real root of the equation $x^3 - 2x - 5 = 0$ correct upto four decimal places.

Ans.: Given equation is

$$f(x) = x^3 - 2x - 5 \qquad \text{\_ \_ \_ (1)}$$

In this method following formula is used :-

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})\ f(x_n)}{[f(x_n) - f(x_{n-1})]} \qquad \_\_\_ (2)$$

Taking x = 1 in equation (1)

$f(1) = 1 - 2 - 5 = -6$ (which is negative)

Taking x = 2 in equation (1)

$f(2) = 8 - 4 - 5 = -1$ (which is negative)

Taking x = 3

$f(3) = 27 - 6 - 5 = 16$ (which is positive)

Since $f(2).f(3) < 0$

So the root of the given equation lies between 2 and 3.

Let $x_1 = 2$ and $x_2 = 3$

$f(x_1) = f(2) = -1$

and $f(x_2) = f(3) = 16$

Now to find x_3 using equation (2)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

$$= 3 - \frac{(3 - 2)}{16 + 1} \times 16$$

$$= 3 - \frac{16}{17} \quad = 2.0588$$

$f(x_3) = (2.0558)^3 - 2\ (2.0588) - 5$

$= 8.7265 - 4.1176 - 5$

$= -0.3911$ (which is negative)

$\therefore \quad f(x_2).f(x_3) < 0$

Now to find x_4 using equation (2)

$$x_4 = x_3 - \frac{(x_3 - x_2)}{[f(x_3) - f(x_2)]} \times f(x_3)$$

$$= 2.0588 - \frac{(2.0588 - 3)}{-0.3911 - 16} \times (-0.3911)$$

$$= 2.0588 + \frac{(-0.9412) \times (-0.3911)}{16.3911} = 2.0812$$

$\therefore$ $f(x_4) = 9.0144 - 4.1624 - 5$

$= -0.148$ (which is negative)

So $f(x_2) . f(x_4) < 0$

Now using equation (2) to find x_5

$$x_5 = x_4 - \frac{(x_4 - x_2)}{[f(x_4) - f(x_2)]} \times f(x_4)$$

$$= 2.0812 - \frac{(2.0812 - 3)}{(-0.148 - 16)} \times (-0.148)$$

$$= 2.0812 + \frac{(-0.9188) \times (-0.148)}{16.148}$$

$$= 2.0812 + 8.4210 \text{ x } \frac{(x_5 - x_2) \times f(x_5)}{f(x_5) - f(x_2)} 10^{-3}$$

$= 2.0896$

$\therefore$ $f(x_5) = 9.1240 - 4.1792 - 5$

$= -0.0552$ (which is negative)

$f(x_2).f(x_5) < 0$

Now using equation (2) to find x_6

$$x_6 = x_5 - \frac{(x_5 - x_2) \times f(x_5)}{f(x_5) - f(x_2)}$$

$$= 2.0896 - \frac{(2.0896 - 3)}{(-0.0552 - 16)} \times (-0.0552)$$

$= 2.0896 + \dfrac{(0.05025)}{16.0552}$

$= 2.0927$

$\therefore \quad f(x_6) = 9.1647 - 4.1854 - 5$

$= -0.0207$ (which is negative)

So $\quad f(x_2).f(x_6) < 0$

Now using equation (2) to find x_7

$$x_7 = x_6 - \frac{(x_6 - x_2)}{f(x_6) - f(x_2)} \times f(x_6)$$

$= 2.0927 - \dfrac{(2.0927 - 3)}{(-0.0207 - 16)} \times (-0.0207)$

$= 2.0927 + \dfrac{(-0.9073)\,(-0.0207)}{16.0207}$

$= 2.0927 + 1.1722 \times 10^{-3}$

$= 2.0938$

Now $\quad f(x_7) = 9.1792 - 4.1876 - 5$

$= -0.0084$ (which is negative)

So $\quad f(x_2).f(x_7) < 0$

Now using equation (2) to find x_8

$$x_8 = x_7 - \frac{(x_7 - x_2)}{f(x_7) - f(x_2)} \times f(x_7)$$

$= 2.0938 - \dfrac{(2.0938 - 3)}{(-0.0084 - 16)} \times (-0.0084)$

$= 2.0938 + \dfrac{(-0.9062)\,(-0.0084)}{16.0084}$

$= 2.0938 + 4.755 \times 10^{-4}$

$= 2.09427$

∴ $f(x_8) = 9.1853 - 4.18854 - 5$

$= -0.00324$ (which is negative)

So $f(x_2).f(x_8) < 0$

Now using equation (2) to find x_9

$$x_9 = x_8 - \frac{(x_8 - x_2)}{f(x_8) - f(x_2)} \times f(x_8)$$

$$= 2.09427 - \frac{(2.09427 - 3)}{(-0.00324 - 16)} \times (-0.00324)$$

$$= 2.09427 - \frac{(-0.90573)\ (-0.00324)}{16.00324}$$

$$= 2.0944$$

∴ The real root of the given equation is 2.094 which is correct upto three decimals.

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Chapter-4

Secant Method

Note : In this method following formula is used to find root –

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})} \qquad \_\_\_ (1)$$

Q.1. Find the root of the equation $x^3 - 5x^2 - 17x + 20$ [use Secant Method] correct upto four decimals.

Ans.: Given $f(x) = x^3 - 5x^2 - 17x + 20$ ___ (2)

Taking x = 0 in equation (1)

$f(0) = 20$

Now taking x = 1

$f(1) = 1 - 5 - 17 + 20$

$= -1$

Since $f(0) = 20$ (positive) and $f(1) = -1$ (which is negative) so the root of the given equation lies between 0 and 1.

Let $x_1 = 0$ and $x_2 = 1$

$\therefore \quad f(x_1) = 20$ and $f(x_2) = -1$

using equation (1) to find x_3

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} \times f(x_2)$$

$= 1 - \frac{(1 - 0)}{(-1) - 20} \times (-1)$

$= 1 + \frac{(1)}{(-21)} = 1 - \frac{1}{21}$

$= 0.9523$

$\therefore \quad f(x_3) = f(0.9523)$

$= (0.9523)^3 - 5\,(0.9523)^2 - 17\,(0.9523) + 20$

$= 0.8636 - 4.5343 - 16.1891 + 20$

$= 0.1402$ (which is positive)

Using equation (1) to find x_4

$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} \times f(x_3)$

$= 0.9523 - \frac{(0.9523 - 1)}{[0.1402 - (-1)]} \times 0.1402$

$= 0.9523 - \frac{(-0.0477)\,(0.1402)}{(1.1402)}$

$= 0.9523 + 0.005865 \quad = 0.9581$

$f(x_4) = (0.9581)^3 - 5\,(0.9581)^2 - 17\,(0.9581) + 20$

$= 0.8794 - 4.5897 - 16.2877 + 20$

$= 0.0020$ (which is positive)

$x_5 = x_4 - \frac{(x_4 - x_3)}{f(x_4) - f(x_3)} \times f(x_4)$

$= 0.9581 - \frac{(0.9581 - 0.9523)}{(0.0020) - (0.1402)} \times 0.0020$

$= 0.9581$

Hence the root of the given equation is 0.9581 which is correct upto four decimal.

Q.2. Given that one of the root of the non-linear equation cos x – xe^x = 0 lies between 0.5 and 1.0 find the root correct upto three decimal places, by Secant Method.

Ans.: Given equation is $f(x) = \cos x - xe^x$

And $x_1 = 0.5$ and $x_2 = 1.0$

$$f(x_1) = \cos(0.5) - (0.5)e^{0.5}$$

$$= 0.87758 - 0.82436$$

$$= 0.05321$$

Now $f(x_2) = \cos(1) - (1)e^1$

$$= 0.54030 - 2.71828$$

$$= -2.1780$$

Now to calculate x_3 using equation (1)

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} \times f(x_2)$$

$$= 1 - \frac{(1-0.5)}{(-2.1780-0.05321)} \times (-2.1780)$$

$$= 1 - \frac{(0.5)(2.1780)}{2.23121}$$

$$= 1 - 0.48807$$

$$= +0.51192$$

$\therefore$ $f(x_3) = f(0.51192)$

$$= \cos(0.51192) - (0.51192)e^{0.51192}$$

$$= 0.87150 - 0.85413$$

$$= 0.01767$$

Now for calculating x_4 using equation (1)

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} \times f(x_3)$$

$$= 0.51192 - \frac{(0.51192-1)}{(0.1767)-(-2.1780)} \times 0.01767$$

$$= 0.51192 - \frac{(-0.48808)(0.01767)}{2.19567}$$

$$= 0.51192 + \frac{0.0086243}{2.19567}$$

$$= 0.51192 + 0.003927$$

$$= 0.51584$$

$\therefore \quad f(x_4) = \cos(0.51584) - (0.51584)e^{0.51584}$

$= 0.86987 - 0.86405$

$= 0.005814$ (which is positive)

Now for calculating x_5 using equation (1)

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f(x_4) - f(x_3)} \times f(x_4)$$

$$= 0.51584 - \frac{(0.51584 - 0.51192)}{(0.005814 - 0.01767)} \times 0.005814$$

$$= 0.51584 - \frac{0.00392}{(-0.01185)} \times (0.005814)$$

$$= 0.51584 + 0.001923$$

$$= 0.51776$$

$$= 0.5178$$

Now $\quad f(x_5) = \cos(0.5178) - (0.5178)e^{0.5178}$

$= 0.8689 - 0.8690$

$= -0.00001$

$= -0.0000$ (upto four decimals)

Hence the root of the given equation is $x = 0.5178$ (which is correct upto four decimal places)

This process cannot be proceed further because $f(x_5)$ vanishes.

□ □ □

Chapter-5

Newton Raphson Method

Hint : Formula uses in this method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q.1. Find the root of the equation $x^2 - 5x + 2 = 0$ correct upto 5 decimal places. (use Newton Raphson Method.)

Ans.: : Given $f(x) = x^2 - 5x + 2 = 0$

Taking x = 0

$f(0) = 2$ (which is positive)

Taking x = 1

$f(1) = 1 - 5 + 2 = -2$ (which is negative)

$f(0) \, . \, f(1) < 0$

∴ The root of the given equation lies between 0 and 1

Taking initial approximation as

$$x_1 = \frac{0+1}{2} = 0.5$$

$f(x) = x^2 - 5x + 2$

$f'(x) = 2x - 5$

Since $x_1 = 0.5$

$f(x_1) = (0.5)^2 - 5(0.5) + 2$

$= 0.25 - 2.5 + 2$

$= - 0.25$

$f'(x_1) = 2(0.5) - 5$

$= 1 - 5$

$= - 4$

Now finding x_2

$$x_2 = 0.5 - \frac{(-0.25)}{-4}$$

$$= 0.5 - \frac{0.25}{4}$$

$= 0.4375$

$f(x_2) = (0.4375)^2 - 5(0.4375) + 2$

$= 0.19140 - 2.1875 + 2$

$= 0.003906$

$f'(x_2) = 2(0.4375) - 5$

$= - 4.125$

Now finding x_3

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)}$$

$$= 0.4375 - \frac{0.003906}{(-4.125)}$$

$= 0.4375 + 0.0009469$

$= 0.43844$

$f(x_3) = (0.43844)^2 - 5(0.43844) + 2$

$= 0.19222 - 2.1922 + 2$

$= 0.00002$

$f'(x_3) = 2 \times (0.43844) - 5$

$= - 4.12312$

$$x_4 = x_3 - \frac{f(x_3)}{f^1(x_3)}$$

$$= 0.43844 - \frac{0.00002}{(-4.12312)}$$

$$= 0.43844 + 0.00000485$$

$$= 0.43844$$

Hence the root of the given equation is 0.43844 which is correct upto five decimal places.

Q.2. Apply Newton Raphson Method to find the root of the equation 3x – cos x – 1 = 0 correct the result upto five decimal places.

Ans.: Given equation is

$$f(x) = 3x - \cos x - 1$$

Taking x = 0

$$f(0) = 3(0) - \cos 0 - 1$$

$$= -2$$

Now taking x = 1

$$f(1) = 3(1) - \cos(1) - 1$$

$$= 3 - 0.5403 - 1$$

$$= 1.4597$$

Taking initial approximation as

$$x_1 = \frac{0+1}{2} = 0.5$$

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

At $x_1 = 0.5$

$$f(x_1) = 3(0.5) - \cos(0.5) - 1$$

$$= 1.5 - 0.8775 - 1$$

$= -0.37758$

$f'(x_1) = 3 - \sin(0.5)$

$= 3.47942$

Now to find x_2 using following formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{(-0.37758)}{(3.47942)}$$

$= 0.5 + 0.10851$

$= 0.60852$

$f(x_2) = 3(0.60852) - \cos(0.60852) - 1$

$= 1.82556 - 0.820494 - 1$

$= 0.005066$

$f'(x_2) = 3 + \sin(0.60852)$

$= 3.57165$

Now finding x_3

$$x_3 = 0.60852 - \frac{(0.005066)}{(3.57165)}$$

$= 0.60852 - 0.0014183$

$= 0.60710$

$f(x_3) = 3(0.60710) - \cos(0.60710) - 1$

$= 1.8213 - 0.821305884 - 1$

$= -0.00000588$

$f'(x_3) = 3 + \sin(0.60710)$

$= 3 + 0.57048$

$= 3.5704$

Now to find x_4 using following formula

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.60710 - \frac{(-0.00000588)}{3.5704}$$

$$= 0.60710 + 0.00000164$$

$$= 0.60710$$

Which is same as x_3

Hence the root of the given equation is x = 0.60710 which is correct upto five decimal places.

□ □ □

Chapter-6

Iterative Method

Q.1. Find a root of the equation $x^3 + x^2 - 1 = 0$ in the interval (0,1) with an accuracy of 10^{-4}.

Ans.: Given equation is $f(x) = x^3 + x^2 - 1 = 0$

Rewriting above equation in the form

$$x = \phi (x)$$

The given equation can be expressed in either of the form :

(i) $x^3 + x^2 - 1 = 0$

$x^3 + x^2 = 1$

$x^2 (x + 1) = 1$

$x^2 = \frac{1}{1 + x}$

$$x = \frac{1}{\sqrt{(1+x)}} \qquad \text{---} (1)$$

(ii) $x^3 + x^2 - 1 = 0$

$x^2 = 1 - x^3$

$$x = (1 + x^3)^{-(1/2)} \qquad \text{---} (2)$$

(iii) $x^3 + x^2 - 1 = 0$

$x^3 = 1 - x^2$

$$x = (1 - x^2)^{1/3} \qquad \text{---} (3)$$

Comparing equation (1) with x – g (x) = 0 we find that

$$g(x) = \frac{1}{\sqrt{(1+x)}}$$

$$g(x) = (1 + x)^{-1/2}$$

$$g'(x) = -½\,(1 + x)^{-3/2}$$

$$|g'(x)| = ½\,(1 + x)^{3/2}$$

$$= \frac{1}{2(1+x)^{3/2}} < 1$$

Now comparing equation (2) with x – g(x) = 0

We find that $g(x) = (1 - x^3)^{1/2}$

$$g'(x) = ½\,(1 + x^3)^{-1/2} \text{ x } (-3x^2)$$

$$= \frac{-3}{2}\,\frac{0+1}{2}$$

$$|g'(x)| = \frac{3}{2}\,\frac{x^2}{(1-x^2)^{1/2}}$$

Which is not less than one.

Now comparing equation (3) with x – g(x) = 0

$$g(x) = (1 - x^2)^{1/3}$$

$$g'(x) = \frac{1}{3}\,(1 - x^2)^{-2/3} \text{ x } (-2x)$$

$$= -\frac{2}{3}\,\frac{x}{(1-x^2)^{1/2}}$$

$$|g'(x)| = \frac{2}{3}\,\frac{x}{(1-x^2)^{2/3}}$$

Which is not less than one.

Hence this method is applicable only to equation (1) because it is convergent for all $x \in (0, 1)$

Now taking initial approximation

$$x_1 = \frac{0+1}{2} = 0.5$$

So $x_2 = \frac{1}{\sqrt{(1+x_1)}}$ [using iteration scheme $x_{n+1} = \frac{1}{\sqrt{(x_n+1)}}$]

$$x_2 = \frac{1}{\sqrt{0.5+1}} = \frac{1}{\sqrt{1.5}} = 0.81649$$

Similarly

$$x_3 = \frac{1}{\sqrt{(x_2+1)}} = \frac{1}{\sqrt{0.81649+1}} = 0.7419$$

$$x_4 = \frac{1}{\sqrt{(x_3+1)}} = \frac{1}{\sqrt{0.7419+1}} = 0.7576$$

$$x_5 = \frac{1}{\sqrt{(x_4+1)}} = \frac{1}{\sqrt{0.7576+1}} = 0.7542$$

$$x_6 = \frac{1}{\sqrt{(x_5+1)}} = \frac{1}{\sqrt{0.7542+1}} = 0.7550$$

$$x_7 = \frac{1}{\sqrt{(x_6+1)}} = \frac{1}{\sqrt{0.7550+1}} = 0.7548$$

$$x_8 = \frac{1}{\sqrt{(x_7+1)}} = \frac{1}{\sqrt{0.7548+1}} = 0.7548$$

Hence the approximate root of the given equation is x = 0.7548

Q.2. Find the root of the equation 2x = cos x + 3 correct upto 3 decimal places.

Ans.: Given equation is

$$f(x) = 2x - \cos x - 3 = 0$$

Rewriting above equation in the form x = g(x)

$$\Rightarrow 2x = \cos x + 3$$

$$\Rightarrow x = \frac{\cos x + 3}{2} \qquad \text{---} (1)$$

Comparing above equation with the following equation x = g(x) we find the

$$g(x) = \frac{\cos x + 3}{2} = \frac{\cos x}{2} + \frac{3}{2}$$

$$g'(x) = \frac{-\sin x}{2}$$

$$|g'(x)| = \frac{\sin x}{2}$$

For $x \in (1, 2)$

$$|\sin x| < 1$$

Hence the iterative scheme $x_{n+1} = \dfrac{\cos (x_n) + 3}{2}$ is convergent.

Now taking initial approximation $x_1 = 1.5$

$$\therefore \quad x_2 = \frac{\cos x_1 + 3}{2} = \frac{\cos (1.5) + 3}{2} = 1.5353$$

$$x_3 = \frac{\cos (x_2) + 3}{2} = \frac{\cos (1.5353) + 3}{2} = 1.5177$$

$$x_4 = \frac{\cos (x_3) + 3}{2} = \frac{\cos (1.5177) + 3}{2} = 1.5265$$

$$x_5 = \frac{\cos (x_4) + 3}{2} = \frac{\cos (1.5265) + 3}{2} = 1.5221$$

$$x_6 = \frac{\cos (x_5) + 3}{2} = \frac{\cos (1.5221) + 3}{2} = 1.5243$$

$$x_7 = \frac{\cos (x_6) + 3}{2} = \frac{\cos (1.5243) + 3}{2} = 1.5230$$

$$x_8 = \frac{\cos (x_7) + 3}{2} = \frac{\cos (1.5230) + 3}{2} = 1.523$$

Which is same as x_7

Hence the root of the given equation is x = 1.523 (which is correct upto 3 decimals)

Q.3. Find the root of the equation $xe^x = 1$ in the internal (0, 1) (use iterative Method)

Ans.: Given equation is $xe^x - 1 = 0$

Rewriting above equation in the form of x = g (x)

$$xe^x - 1 = 0$$

$$xe^x = 1$$

$$x = e^{-x}$$

Comparing it with the equation x = g (x) we find that

$$g(x) = e^{-x}$$

$$g'(x) = -e^{-x}$$

$$|g'(x)| = e^{-x} < 1$$

Hence the iterative scheme is

$$x_{n+1} = e^{-x_n}$$

Now taking initial approximation

$$x_1 = 0.5$$

$$x_2 = e^{-x_1} = e^{-(0.5)} = 0.60653$$

$$x_3 = e^{-x_2} = e^{-(0.6065)} = 0.5452$$

$$x_4 = e^{-x_3} = e^{-(0.5452)} = 0.5797$$

$$x_5 = e^{-x_4} = e^{-0.5797} = 0.5600$$

$$x_6 = e^{-x_5} = e^{-0.5600} = 0.5712$$

$$x_7 = e^{-x_6} = e^{-(0.5712)} = 0.5648$$

$$x_8 = e^{-x_7} = e^{-(0.5648)} = 0.5684$$

$$x_9 = e^{-x_8} = e^{-(0.5684)} = 0.5664$$

$$x_{10} = e^{-x_9} = e^{-(0.5664)} = 0.5675$$

Now

$$x_{11} = e^{-x_{10}} = e^{-0.5675} = 0.5669$$

$$x_{12} = e^{-x_{11}} = e^{-0.5669} = 0.5672$$

$$x_{13} = e^{-x_{12}} = e^{-(0.5672)} = 0.5671$$

$$x_{14} = e^{-x_{13}} = e^{-(0.5671)} = 0.5671$$

Hence the approximate root the given equation is x = 0.5671

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Chapter-7

Gauss Elimination Method

Q.1. Use gauss elimination method to solve :

x + y + z = 7

3x + 3y + 4z = 24

2x + y + 3z = 16

Ans.: Since in the first column the largest element is 3 in the second equation, so interchanging the first equation with second equation and making 3 as first pivot.

$3x + 3y + 4z = 24$ _ _ _ (1)

$x + y + z = 7$ _ _ _ (2)

$2x + y + 3z = 16$ _ _ _ (3)

Now eliminating x form equation (2) and equation (3) using equation (1)

-3 × equation (2) + 2 × equation (1), 3 × equation (3) – 2 × equation (1)

we get

$$\begin{array}{r} -3x - 3y - 3z = -21 \\ 3x + 3y + 4z = 24 \\ \hline z = 3 \end{array}$$

and

$$\begin{array}{r} 6x + 3y + 9z = 48 \\ 6x + 6y + 8z = 48 \\ - \quad - \quad - \quad - \\ \hline -3y + z = 0 \end{array}$$

$= 3y - z = 0$

$3x + 3y + 4z = 24$ _ _ _ (4)

$z = 3$ _ _ _ (5)

$3y – z = 0$ _ _ _ (6)

Now since the second row cannot be used as the pivot row since $a_{22} = 0$ so interchanging the equation (5) and (6) we get

$3x + 3y + 4z = 24$ ___ (7)

$3y - z = 0$ ___ (8)

$z = 3$ ___ (9)

Now it is upper triangular matrix system. So by back substitution we obtain.

$\boxed{z = 3}$

From equation (8)

$3y - 3 = 0$

$3y = 3$

$\boxed{y = 1}$

From equation (7)

$3x + 3(1) + 4(3) = 24$

$3x + 3 + 12 = 24$

$3x + 15 = 24$

$3x = 9$

$\boxed{x = 3}$

Hence the solution fo given system of linear equation is

$x = 3$, $y = 1$, $z = 3$

Q.2. Solve the following system of linear equation by Gauss Elimination Method :

$$2x_1 + 4x_2 + x_3 = 3$$

$$3x_1 + 2x_2 - 2x_3 = -2$$

$$x_1 - x_2 + x_3 = 6$$

Ans.: Since in the first column the largest element is 3 in the second row, so interchanging first equation with second equation and making 3 as first pivot.

$3x_1 + 4x_2 - 2x_3 = -2$ ___ (1)

$2x_1 + 4x_2 + x_3 = 3$ ___ (2)

$x_1 - x_2 + x_3 = 6$ ___ (3)

Eliminating x_1 form equation (2) and equation (3) using equation (1)

-3 × equation (2) + 2 × equation (1) and + 3 × equation (3) – equation (1)

$$\begin{array}{r} -6x_1 - 12x_2 - 3x_3 = -9 \\ 6x_1 + 4x_2 - 4x_3 = -4 \\ \hline -8x_2 - 7x_3 = -13 \end{array} \quad \text{and} \quad \begin{array}{r} 3x_1 - 3x_2 + 3x_3 = 18 \\ 3x_1 + 2x_2 - 2x_3 = -2 \\ - \quad - \quad + \quad + \\ \hline -5x_2 + 5x_3 = 20 \end{array}$$

$8x_2 + 7x_3 = 13$ $x_2 - x_3 = -4$

So the system now becomes :

$3x_1 + 2x_2 - 2x_3 = -2$ ___ (4)

$8x_2 + 7x_3 = 13$ ___ (5)

$x_2 - x_3 = -4$ ___ (6)

Now eliminating x_2 from equation (6) using equation (5) {8 × equation (6) –equation (5)}

$$\begin{array}{r} 8x_2 - 8x_3 = -32 \\ 8x_2 + 7x_3 = -13 \\ - \quad - \quad + \\ \hline -15x_3 = -45 \end{array}$$

$x_3 = 3$

So the system of linear equation is

$3x_1 + 2x_2 - 2x_3 = -2$ ___ (7)

$8x_2 + 7x_3 = 13$ ___ (8)

$x_3 = 3$ ___ (6)

Now it is upper triangular system so by back substitution we obtain

$x_3 = 3$

From equation (8)

$8x_2 + 7(3) = 13$

$8x_2 = 13 - 21$

$8x_2 = -8$

$x_2 = -1$

From equation (9)

$3x_1 + 2(-1) - 2(3) = -2$

$3x_1 = -2 + 2 + 6$

$3x_1 = 6$

$x_1 = 2$

∴ Hence the solution of the given system of linear equation is :

$x_1 = 2$, $x_2 = -1$, $x_3 = 3$

□ □ □

Chapter-8

Gauss-Jordan Elimination Method

Q.1. Solve the following system of equations :

$$10x_1 + 2x_2 + x_3 = 9 \quad \text{---} (1)$$

$$2x_1 + 20x_2 - 2x_3 = -44 \quad \text{---} (2)$$

$$-2x_1 + 3x_2 + 10x_3 = 22 \quad \text{---} (3)$$

Use Gauss Jordan Method.

Ans.: Since in the given system pivoting is not necessary. Eliminating x_1 from equation (2) and equation (3) using equation (1)

5 × equation (2) – equation (1) , 5 × equation (3) **+** equation (1)

$$\begin{array}{r} 10x_1 - 100x_2 - 10x_3 = -220 \\ \underline{-10x_1 + 2x_2 + x_3 \quad = \ -9} \\ 98x_2 - 11x_3 = -229 \end{array} \quad \text{and} \quad \begin{array}{r} -10x_1 + 15x_2 + 50x_3 = 110 \\ \underline{10x_1 + 2x_2 + x_3 \quad = \quad 9} \\ 17x_2 + 51x_3 = 119 \\ = x_2 + 3x_3 = 7 \end{array}$$

Now the system of equation becomes

$$10x_1 + 2x_2 + x_3 = 9 \quad \text{---} (4)$$

$$98x_2 - 11x_3 = -229 \quad \text{---} (5)$$

$$x_2 + 3x_3 = 7 \quad \text{---} (6)$$

Now eliminating x_2 from equation (4) and (6) using equation (5)

98 × equation (6) – equation (5) , 49 × equation (4) - equation (5)

$$\begin{array}{r} 98x_2 + 294x_3 = 686 \\ \underline{\underset{-}{}98x_2 \underset{+}{-} 11x_3 = \underset{+}{-} 229} \\ 305x_3 = 915 \end{array}$$

$$x_3 = 3$$

$$\begin{array}{r} 490x_1 + 98x_2 + 49x_3 = 441 \\ \underline{\underset{-}{}98x_2 \underset{+}{-} 11x_3 \underset{+}{=} 9} \\ 490x_1 + 60x_3 = 670 \end{array}$$

$$= 49x_1 + 6x_3 = 67$$

Now the system of equation becomes :

$49x_1 + 0 + 6x_3 = 67$ _ _ _ (7)

$98x_2 - 11x_3 = -229$ _ _ _ (8)

$x_3 = 3$ _ _ _ (9)

Hence it reduces to upper triangular system now by back substitution.

$x_3 = 3$

From equation (8)

$98x_2 - 11 \times 3 = -229$

$98x_2 = -229 + 33$

$98x_2 = -196$

$x_2 = -2$

From equation (7)

$49x_1 + 6(3) = 67$

$49x_1 = 67 - 18$

$49x_1 = 49$

$x_1 = 1$

Thus the solution of the given system of linear equation is

$x_1 = 1$, $x_2 = -2$, $x_3 = 3$

Q.2. Solve the following system of equation using Gauss-Jordan Elimination Method.

$\mathbf{2x_1 - 2x_2 + 5x_3 = 13}$ _ _ _ **(1)**

$\mathbf{2x_1 + 3x_2 + 4x_3 = 20}$ _ _ _ **(2)**

$\mathbf{3x_1 - x_2 + 3x_3 = 10}$ _ _ _ **(3)**

Ans.: Solve this question like question no. 17.

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Chapter-9

Matrix Inversion Method

Q.1. Solve the given system of equation using Matrix inversion Method.

$$6x_1 + 3x_2 + 7x_3 = 7$$

$$x_1 + 5x_2 + 2x_3 = -7$$

$$7x_1 + 2x_2 + 10x_3 = 13$$

Ans.: The given system of equations cab be written in the form of AX = B

$$A = \begin{bmatrix} 6 & 3 & 7 \\ 1 & 5 & 2 \\ 7 & 2 & 10 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ -7 \\ 13 \end{bmatrix}$$

The solution can be given by $X = A^{-1}B$ so to find the solution first we have to find A^{-1} using Gauss-Jordan Method. The inverse of matrix A that is A^{-1} is obtained by reducing the argumented matrix [A/I] into the matrix $[I/A^{-1}]$

The argumented matrix is given by

$$\left[\begin{array}{ccc} 6 & 3 & 7 \\ 1 & 5 & 2 \\ 7 & 2 & 10 \end{array} \middle/ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|ccc} 7 & 2 & 10 & 0 & 0 & 1 \\ 1 & 5 & 2 & 0 & 1 & 0 \\ 6 & 3 & 7 & 1 & 0 & 0 \end{array}\right]$$

$R_2 \rightarrow 7R_2 - R_1$, $R_3 \rightarrow \frac{7}{6} R_3 - R_1$

$$\left[\begin{array}{ccc|ccc} 7 & 2 & 10 & 0 & 0 & 1 \\ 0 & 33 & 4 & 0 & 7 & -1 \\ 0 & \frac{3}{2} & -\frac{11}{6} & \frac{7}{6} & 0 & -1 \end{array}\right]$$

$R_1 \rightarrow -\frac{33}{2} R_1 + R_2$, $R_3 \rightarrow \frac{-2 \text{ x } 33}{3} R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} \frac{-231}{2} & 2 & -166 & 0 & 7 & -\frac{35}{2} \\ 0 & 33 & 4 & 0 & 7 & -1 \\ 0 & 0 & \frac{133}{3} & \frac{-77}{3} & 0 & 21 \end{array}\right]$$

$R_1 \rightarrow \frac{133 \, R_1}{2 \text{ x } 161} + R_3$, $R_2 \rightarrow \frac{-133}{3 \text{ x } 4} R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} \frac{-10241}{322} & 0 & 0 & \frac{-77}{3} & \frac{616}{69} & \frac{2233}{138} \\ 0 & \frac{-1463}{4} & 0 & \frac{-77}{3} & \frac{-847}{12} & \frac{385}{12} \\ 0 & 0 & \frac{133}{3} & \frac{-77}{3} & 7 & 21 \end{array}\right]$$

$R_1 \leftrightarrow \frac{-322}{-10241} R_1$, $R_2 \leftrightarrow \frac{4}{-1463} R_2$, $R_3 \leftrightarrow \frac{3}{133} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{46}{57} & \frac{-16}{57} & \frac{-29}{57} \\ 0 & 1 & 0 & \frac{4}{57} & \frac{11}{57} & \frac{-5}{57} \\ 0 & 0 & 1 & \frac{-11}{19} & \frac{3}{19} & \frac{9}{19} \end{array}\right]$$

$R_1 \leftrightarrow \frac{-322}{-10241} R_1$, $R_2 \leftrightarrow \frac{4}{-1463} R_2$, $R_3 \leftrightarrow \frac{3}{133} R_3$

Hence

$$A^{-1} = \frac{1}{57}\begin{bmatrix} 46 & -16 & -29 \\ 4 & 11 & -5 \\ -33 & 9 & 27 \end{bmatrix}$$

Thus the matrix A is reduced to identity matrix Hence the solution of the given system of equations is

$X = A^{-1} B$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{57}\begin{bmatrix} 46 & -16 & -29 \\ 4 & 11 & -5 \\ -33 & 9 & 27 \end{bmatrix}\begin{bmatrix} 7 \\ -7 \\ 13 \end{bmatrix}$$

$$= \frac{1}{57}\begin{bmatrix} 322 + 112 - 377 \\ 28 - 77 - 65 \\ 231 - 63 + 351 \end{bmatrix} = \frac{1}{57}\begin{bmatrix} 57 \\ -114 \\ 57 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Q.2. Solve the following system of linear equations using matrix inversion method.

$\mathbf{3x_1 + 2x_2 + 4x_3 = 7}$

$\mathbf{2x_1 + x_2 + x_3 = 7}$

$\mathbf{x_1 + 3x_2 + 4x_3 = 2}$

Ans.: The given system of linear equations can be written in the form of AX = B

$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 2 \end{bmatrix}$$

The solution can be given by X = A^{-1} B. for this we have to first find the value of A^{-1} using Gauss Jordan Method.

The inverse of the matrix A using Gauss Jordan method is obtained by reducing the argumented matrix [A/I] in the form of[I/A^{-1}].

The argumented matrix is given as follows :

$$\left[\begin{matrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 4 \end{matrix} \middle/ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]$$

Here pivoting is not necessary.

$R_2 \to \frac{3}{2} R_2 - R_1$, $R_3 = 3R_3 - R_1$

$$= \left[\begin{matrix} 3 & 2 & 4 \\ 0 & \frac{-1}{2} & \frac{-5}{2} \\ 0 & 7 & 8 \end{matrix} \middle/ \begin{matrix} 1 & 0 & 0 \\ -1 & \frac{3}{2} & 0 \\ -1 & 0 & 3 \end{matrix}\right]$$

$R_1 \to \frac{1}{4} R_1 + R_2$, $R_3 = \frac{1}{14} R_3 + R_2$

$$= \left[\begin{matrix} \frac{3}{4} & 0 & \frac{-3}{2} \\ 0 & \frac{-1}{2} & \frac{-5}{2} \\ 0 & 0 & \frac{-27}{14} \end{matrix} \middle/ \begin{matrix} \frac{-3}{4} & \frac{3}{2} & 0 \\ -1 & \frac{3}{2} & 0 \\ \frac{-15}{14} & \frac{3}{2} & \frac{3}{14} \end{matrix}\right]$$

$R_1 \to \frac{-9}{7} R_1 + R_3$, $R_2 \to \frac{-27}{35} R_2 + R_3$

$$= \left[\begin{array}{ccc|ccc} \frac{-27}{28} & 0 & 0 & \frac{-3}{28} & \frac{-3}{7} & \frac{3}{14} \\ 0 & \frac{27}{70} & 0 & \frac{-3}{10} & \frac{12}{35} & \frac{3}{14} \\ 0 & 0 & \frac{-27}{14} & \frac{-15}{14} & \frac{3}{2} & \frac{3}{14} \end{array}\right]$$

Now $R_1 \to \frac{-28}{27} R_1$, $R_2 \to \frac{70}{27} R_2$, $R_3 \to \frac{-14}{27} R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{4}{9} & \frac{-2}{9} \\ 0 & 1 & 0 & \frac{-7}{9} & \frac{8}{9} & \frac{5}{9} \\ 0 & 0 & 1 & \frac{5}{9} & \frac{-7}{9} & \frac{-1}{9} \end{array}\right]$$

Hence

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 4 & -2 \\ -7 & 8 & 5 \\ 5 & -7 & -1 \end{bmatrix}$$

Thus the solution of given matrix is given by

$$X = A^{-1} B$$

i.e. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 4 & -2 \\ -7 & 8 & 5 \\ 5 & -7 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 7 & +28 & -4 \\ -49 & +56 & +10 \\ 35 & -49 & -2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 31 \\ 17 \\ -16 \end{bmatrix}$$

Hence $x_1 = \frac{31}{9}$, $x_2 = \frac{17}{9}$, $x_3 = \frac{-16}{9}$

□ □ □

Chapter-10

Matrix Factorization Method

Q.1. Solve the following system of linear equation using Matrix Factorization Method.

$$\mathbf{3x_1 + 5x_2 + 2x_3 = 8}$$

$$\mathbf{8x_2 + 2x_3 = -7}$$

$$\mathbf{6x_1 + 2x_2 + 8x_3 = 26}$$

Ans.: Above system of equation can be written in the form AX = B where

$$A = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix}; \quad B = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let us assume that A = LU

$$\text{Where } L = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{l}_{21} & 1 & 0 \\ \mathbf{l}_{31} & \mathbf{l}_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\therefore \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{l}_{21} & 1 & 0 \\ \mathbf{l}_{31} & \mathbf{l}_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ \mathbf{l}_{21}U_{11} & \mathbf{l}_{21}U_{12} + U_{22} & \mathbf{l}_{21}U_{13} + U_{23} \\ \mathbf{l}_{31}U_{11} & \mathbf{l}_{31}U_{12} + \mathbf{l}_{32}U_{22} & \mathbf{l}_{31}U_{13} + \mathbf{l}_{32}U_{23} + U_{33} \end{bmatrix}$$

Since A = LU so comparing both matrices.

$U_{11} = 3$ _ _ _ (1)

$U_{12} = 5$ _ _ _ (2)

$U_{13} = 2$ _ _ _ (3)

$l_{21}U_{11} = 0$ _ _ _ (4)

$l_{21}U_{12} + U_{22} = 8$ _ _ _ (5)

$l_{21}U_{13} + U_{23} = 2$ _ _ _ (6)

$l_{31}U_{11} = 6$ _ _ _ (7)

$l_{31}U_{12} + l_{32}U_{22} = 2$ _ _ _ (8)

$l_{31}U_{13} + l_{32}U_{22} + U_{33} = 8$ _ _ _ (9)

$l_{21}U_{11} = 0$

$\Rightarrow l_{21} \times = 0$

$\Rightarrow l_{21} = 0$ _ _ _ (10)

$l_{31}U_{11} = 6$

$\Rightarrow l_{31} \times 3 = 6$

$\Rightarrow l_{31} = 6/3$

$\Rightarrow l_{31} = 2$ _ _ _ (11)

Now from equation (5)

$l_{21}U_{12} + U_{22} = 8$

$\Rightarrow 0 \times U_{12} + U_{22} = 8$

$\Rightarrow U_{22} = 8$ _ _ _ (12)

From equation (6)

$l_{21}U_{13} + U_{23} = 2$

$\Rightarrow 0 \times U_{13} + U_{23} = 2$

$\Rightarrow U_{23} = 2$ _ _ _ (13)

From equation (8)

$l_{31}U_{12} + \square_{32}U_{22} = 2$

$\Rightarrow 2 \times 5 + L_{32} \times 8 = 2$

$\Rightarrow l_{32} \times 8 = 2 - 10 = -8$

$\Rightarrow l_{32} = -1$

From equation (9)

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 8$

$\Rightarrow 2 \times 2 + (-1) \times 2 + U_{33} = 2$

$\Rightarrow U_{33} = 8 - 4 + 2$

$\Rightarrow U_{33} = 6$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Since the given system of equation can be written as AX = B [Here A = LU]

$\therefore \quad LUX = B$ _ _ _ (14)

Now let UX = Y _ _ _ (15)

$\therefore \quad LY = B$ _ _ _ (16)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ 2y_1 - y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

On comparing both matrices, we get

$y_1 = 8$, $y_2 = -7$

and $2y_1 - y_2 + y_3 = 26$

$\Rightarrow 2 \times 8 + 7 + y_3 = 26$

$\Rightarrow y_3 = 26 - 16 - 7$

$\Rightarrow y_3 = 3$

$$\therefore \quad Y = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

From equation (15)

UX =Y

$$\Rightarrow \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_1 & + & 5x_2 & + & 2x_3 \\ 0 & + & 8x_2 & + & 2x_3 \\ 0 & + & 0 & + & 6x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

Comparing both matrices

$3x_1 + 5x_2 + 2x_3 = 8$ _ _ _ (17)

$8x_2 + 2x_3 = -7$ _ _ _ (18)

$6\, x_3 = 3$ _ _ _ (19)

From equation (19)

$6x_3 = 3$

$\Rightarrow x_3 = \frac{1}{2}$

From equation (18)

$8x_2 + 2x_3 = -7$

$\Rightarrow 8x_2 + 2 \times \frac{1}{2} = -7$

$\Rightarrow 8x_2 = -7 - 1$

$\Rightarrow 8x_2 = -8$

$\Rightarrow x_2 = -1$

From equation (17)

$3x_1 + 5x_2 + 2x_3 = 8$

$\Rightarrow 3x_1 = 8 + 5 - 1$

$\Rightarrow 3x_1 = 12$

$\Rightarrow x_1 = 4$

Thus the solution of given system of equation is

$x_1 = 4$, $x_2 = -1$ and $x_3 = \frac{1}{2}$

Q. 2. Solve the following system of linear equation using Matrix Factorization Method.

x + 2y + 3z = 14

3x + y + 2z = 11

2x + 3y + z = 11

Ans.: Above system of equation can be written in the form of AX = B where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let us assume that A = LU where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\Rightarrow LU = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21}U_{13} + U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + U_{33} \end{bmatrix}$$

Since A = LU so comparing both sides we get

$U_{11} = 1$ ――― (1)

$U_{12} = 2$ ――― (2)

$U_{13} = 3$ _ _ _ (3)

$l_{21}U_{11} = 3$ _ _ _ (4)

$l_{21}U_{12} + U_{22} = 1$ _ _ _ (5)

$l_{21}U_{13} + U_{23} = 2$ _ _ _ (6)

$l_{31}U_{11} = 2$ _ _ _ (7)

$l_{31}U_{12} + l_{32}U_{22} = 3$ _ _ _ (8)

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 1$ _ _ _ (9)

From equation (3)

$l_{21}U_{11} = 3$

$\Rightarrow l_{21} \times 1 = 3$

$\Rightarrow l_{21} = 3$ _ _ _ (10)

From equation (5)

$l_{21}U_{12} + U_{22} = 1$

$\Rightarrow 3 \times 2 + U_{22} = 1$

$\Rightarrow U_{22} = 1 - 6$

$\Rightarrow U_{22} = -5$ _ _ _ (11)

From equation (6)

$l_{21}U_{13} + U_{23} = 2$

$\Rightarrow 3 \times 3 + U_{23} = 2$

$\Rightarrow U_{23} = 2 - 9$

$\Rightarrow U_{23} = -7$ _ _ _ (12)

From equation (7)

$l_{31}U_{11} = 2$

$\Rightarrow l_{31} \times 1 = 2$

$\Rightarrow l_{31} = 2$ _ _ _ (13)

From equation (8)

$l_{31}U_{12} + l_{32}U_{22} = 3$

$\Rightarrow 2 \times 2 + l_{32}(-5) = 3$

$\Rightarrow l_{32} \times (-5) = 3 - 4$

$\Rightarrow l_{32} = \dfrac{1}{5}$ ––– (14)

From equation (9)

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 1$

$\Rightarrow 2 \times 3 + \dfrac{1}{5} \times (-7) + U_{33} = 1$

$\Rightarrow 6 - \dfrac{7}{5} + U_{33} = 1$

$\Rightarrow U_{33} = 1 - 6 + \dfrac{7}{5} = -5 + \dfrac{7}{5} = \dfrac{-25+7}{5}$

$\Rightarrow U_{33} = \dfrac{-18}{5}$ ––– (15)

$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{5} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & \frac{-18}{5} \end{bmatrix}$

We know that AX = B

$\Rightarrow LUX = B$ where [A = LU] ––– (16)

Now let UX = Y ––– (17)

So LY = B ––– (18)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ 3y_1 + y_2 \\ 2y_1 + \frac{1}{5}y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 11 \\ 11 \end{bmatrix}$$

Comparing both sides we get

$$y_1 = 14 \qquad \text{---} (19)$$

$$3y_1 + y_2 = 11 \qquad \text{---} (20)$$

$$2y_1 + \frac{1}{5} y_2 + y_3 = 11 \qquad \text{---} (21)$$

From equation (20)

$$3y_1 + y_2 = 11$$

$$\Rightarrow 3 \times 14 + y_2 = 11$$

$$\Rightarrow y_2 = 11 - 42$$

$$\Rightarrow y_2 = -31$$

From equation (21)

$$\Rightarrow 2(14) + \frac{1}{5}(-31) + y_3 = 11$$

$$\Rightarrow 28 - \frac{31}{5} + y_3 = 11$$

$$\Rightarrow y_3 = 11 - 28 + \frac{31}{5}$$

$$\Rightarrow y_3 = -17 + \frac{31}{5}$$

$$\Rightarrow y_3 = \frac{-85+31}{5}$$

$$\Rightarrow y_3 = \frac{-54}{5}$$

$$\therefore \quad Y = \begin{bmatrix} 14 \\ -31 \\ \frac{-54}{5} \end{bmatrix}$$

Since UX = Y

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & \frac{-18}{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -31 \\ \frac{-54}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x + 2y + 3z \\ 0 - 5y + 7z \\ 0 + 0 - \frac{18}{5}z \end{bmatrix} = \begin{bmatrix} 14 \\ -31 \\ \frac{-54}{5} \end{bmatrix}$$

$$x + 2y + 3z = 14 \qquad \text{---(22)}$$

$$-5y - 7z = -31 \qquad \text{---(23)}$$

$$\frac{-18}{5}z = \frac{-54}{5} \qquad \text{---(24)}$$

$$\Rightarrow z = \frac{54}{5} \times \frac{5}{18}$$

$$\Rightarrow z = 3$$

From equation (23)

$$5y + 7z = 31$$

$$\Rightarrow 5y + 7 \times 3 = 31$$

$$\Rightarrow 5y = 31 - 21 = 10$$

$$\Rightarrow y = 2$$

From equation (22)

$$x + 2y + 3z = 14$$

$$\Rightarrow x + 4 + 9 = 14$$

$$\Rightarrow x = 14 - 13$$

$$\Rightarrow x = 1$$

Thus the solution of the given system of equation is

x = 1 ; y = 2 and z = 3

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Chapter-11

Jacobi Method

Q.1. Solve the following system of equation by Jacobi Method.

$$83x_1 + 11x_2 - 4x_3 = 95$$

$$7x_1 + 52x_2 + 13x_3 = 104$$

$$3x_1 + 8x_2 + 29x_3 = 71$$

Ans.: Since the given system of equation is

$$83x_1 + 11x_2 - 4x_3 = 95 \qquad \text{\_\_\_ (1)}$$

$$7x_1 + 52x_2 + 13x_3 = 104 \qquad \text{\_\_\_ (2)}$$

$$3x_1 + 8x_2 + 29x_3 = 71 \qquad \text{\_\_\_ (3)}$$

The diagonal elements in the given system of linear equations is not zero so the equation (1), (2) and (3) can be written as :

$$x_1^{(n+1)} = \frac{1}{83}[95 - 11\,x_2^{(n)} + 4\,x_3^{(n)}]$$

$$x_2^{(n+1)} = \frac{1}{52}[104 - 7\,x_1^{(n)} - 13\,x_3^{(n)}] \text{ and}$$

$$x_3^{(n+1)} = \frac{1}{29}[71 - 3\,x_1^{(n)} - 8\,x_2^{(n)}]$$

Now taking initial approximation as :

$$x_1^{(0)} = 0 \quad ; \quad x_2^{(0)} = 0 \quad \text{and} \quad x_3^{(0)} = 0$$

Now for first approximation :

$$x_1^{(1)} = \frac{1}{83}[95 - 11x_2^{(0)} + 4x_3^{(0)}] = 1.1446$$

$$x_2^{(1)} = \frac{1}{52}[104 - 7x_1^{(0)} - 13x_3^{(0)}] = 2$$

$$x_3^{(1)} = \frac{1}{29}[71 - 3x_1^{(0)} - 8x_2^{(0)}] = 2.4483$$

Similarly second approximation :

$$x_1^{(2)} = \frac{1}{83}[95 - 11x_2^{(1)} + 4x_3^{(1)}]$$

$$= \frac{1}{83}[95 - 11(2) + 4(2.4483)] = 0.9975$$

$$x_2^{(2)} = \frac{1}{52}[104 - 7x_1^{(1)} - 13x_3^{(1)}]$$

$$= \frac{1}{52}[104 - 7(1.1446) - 13(2.4483)] = 1.2338$$

$$x_3^{(2)} = \frac{1}{29}[71 - 3x_1^{(1)} - 8x_2^{(1)}]$$

$$= \frac{1}{29}[71 - 3(1.1446) - 8 \times 2] = 1.7781$$

Now the third iteration :

$$x_1^{(3)} = \frac{1}{83}[95 - 11x_2^{(2)} + 4x_3^{(2)}]$$

$$= \frac{1}{83}[95 - 11\times(1.2338) + 4(1.7781)] = 1.0668$$

$$x_2^{(3)} = \frac{1}{52}[104 - 7x_1^{(2)} - 13x_3^{(2)}]$$

$$= \frac{1}{52}[104 - 7\times(0.9975) - 13\times(1.7781)] = \frac{1}{52}[73.9022]$$

$$= 1.4212$$

$$x_3^{(3)} = \frac{1}{29}[71 - 3x_1^{(2)} - 8x_2^{(2)}]$$

$$= \frac{1}{29}[71 - 3 \times (0.9975) - 8 \times (1.2338)] = 2.0047$$

Similarly other iterations are :

$x_1^{(4)} = 1.0528$

$x_2^{(4)} = 1.3552$

$x_3^{(4)} = 1.9459$

$x_1^{(5)} = 1.0588$

$x_2^{(5)} = 1.3718$

$x_3^{(5)} = 1.9655$

$x_1^{(6)} = 1.0575$

$x_2^{(6)} = 1.3661$

$x_3^{(6)} = 1.9603$

$x_1^{(7)} = 1.0580$

$x_2^{(7)} = 1.3676$

$x_3^{(7)} = 1.9620$

$x_1^{(8)} = 1.0579$

$x_2^{(8)} = 1.3671$

$x_3^{(8)} = 1.9616$

$x_1^{(9)} = 1.0579$

$x_2^{(9)} = 1.3671$

$x_3^{(9)} = 1.9616$

Thus the values obtained by successive iteration is given by following table :

x	$x_1^{(n)}$	$x_2^{(n)}$	$x_3^{(n)}$	$x_1^{(n+1)}$	$x_2^{(n+1)}$	$x_3^{(n+1)}$
0	0	0	0	1.1446	2	2.4483
1	1.1446	2	2.4483	0.9975	1.2338	1.7781
2	0.9975	1.2338	1.7781	1.0667	1.4211	2.0047
3	1.0667	1.4211	2.0047	1.0528	1.3552	1.9459
4	1.0528	1.3552	1.9459	1.0587	1.3718	1.9655
5	1.0587	1.3718	1.9655	1.0575	1.3661	1.9603
6	1.0575	1.3661	1.9603	1.0580	1.3676	1.9620
7	1.0580	1.3676	1.9620	1.0579	1.3671	1.9616
8	1.0579	1.3671	1.9616	1.0579	1.3671	1.9616

Thus the solution is

$$x_1 = 1.0579 \quad ; \quad x_2 = 1.3671 \quad \text{and} \quad x_3 = 1.9616$$

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Chapter-12

Gauss Seidel Method

[This method is also called the method of successive displacement]

Q.1. Solve the following linear equation :

$$\mathbf{2x_1 - x_2 + x_3 = 5}$$

$$\mathbf{x_1 + 2x_2 + 3x_3 = 10}$$

$$\mathbf{x_1 + 3x_2 - 2x_3 = 7}$$

(Use Gauss Seidel Method)

Ans.: Above system of equations can be written as :

$$2x_1 - x_2 + x_3 = 5 \qquad \text{\_ \_ \_ (1)}$$

$$x_1 + 3x_2 - 2x_3 = 7 \qquad \text{\_ \_ \_ (2)}$$

$$x_1 + 2x_2 + 3x_3 = 10 \qquad \text{\_ \_ \_ (3)}$$

Iterative equations are :

$$x_1^{(n+1)} = \frac{1}{2}[5 + x_2^{(n)} - x_3^{(n)}] \qquad \text{\_ \_ \_ (4)}$$

$$x_2^{(n+1)} = \frac{1}{3}[7 - x_1^{(n+1)} + 2x_3^{(n)}] \qquad \text{\_ \_ \_ (5)}$$

$$x_3^{(n+1)} = \frac{1}{3}[10 - x_1^{(n+1)} - 2x_2^{(n+1)}] \qquad \text{\_ \_ \_ (6)}$$

Taking initial approximations as :

$x_1^{(0)} = 0$; $x_2^{(0)} = 0$ and $x_3^{(0)} = 0$

First approximation is :

$$x_1^{(1)} = \frac{1}{2}[5 + x_2^{(0)} - x_3^{(0)}]$$

$$= \frac{1}{2}[5 + 0 - 0] = \frac{5}{2} = 2.5$$

$$x_2^{(1)} = \frac{1}{3}[7 - x_1^{(1)} + 2x_3^{(0)}]$$

$$= \frac{1}{3}[7 - 2.5 + 2 \times 0] = \frac{1}{3}(4.5) = 1.5$$

$$x_3^{(1)} = \frac{1}{3}[10 - x_1^{(1)} - 2x_2^{(1)}]$$

$$= \frac{1}{3}[10 - 2.5 - 2 \times 1.5] = 1.5$$

Now second approximation :

$$x_1^{(2)} = \frac{1}{2}[5 + x_2^{(1)} - x_3^{(1)}]$$

$$= \frac{1}{2}[5 + (1.5) - 1.5] = 2.5$$

$$x_2^{(2)} = \frac{1}{3}[7 - x_1^{(2)} + 2x_3^{(1)}]$$

$$= \frac{1}{3}[7 - 2.5 + 2(1.5)] = 2.5$$

$$x_3^{(2)} = \frac{1}{3}[10 - x_1^{(2)} - 2x_2^{(2)}]$$

$$= \frac{1}{3}[10 - 2.5 - 2 \times 2.5] = 0.8333$$

$$x_1^{(3)} = \frac{1}{2}[5 + x_2^{(2)} - x_3^{(2)}]$$

$$= \frac{1}{2}[5 + 2.5 - 0.8333] = 3.3333$$

$$x_2^{(3)} = \frac{1}{3}[7 - x_1^{(3)} + 2x_3^{(2)}]$$

$$= \frac{1}{3}[7 - 3.3333 + 2 \times 0.8333] = 1.7777$$

$$x_3^{(3)} = \frac{1}{3}[10 - x_1^{(3)} - 2x_2^{(3)}]$$

$$= \frac{1}{3}[10 - 3.3333 - 2 \times 1.7777] = 1.0371$$

$\therefore$ $x_1^{(3)} = 3.3333$, $x_2^{(3)} = 1.7777$, $x_3^{(3)} = 1.0371$

$$x_1^{(4)} = \frac{1}{2}[5 + x_2^{(3)} - x_3^{(3)}]$$

$$= \frac{1}{2}[5 + 1.7777 - 1.0371] = 2.8703$$

$x_2^{(4)} = 2.0679$

$x_3^{(4)} = 0.9980$

$\therefore$ $x_1^{(4)} = 2.8703$, $x_2^{(4)} = 2.0679$, $x_3^{(4)} = 0.9980$

Now $x_1^{(5)} = 3.035$

$x_2^{(5)} = 1.9870$

$x_3^{(5)} = 0.9970$

$x_1^{(6)} = 2.9950$

$x_2^{(6)} = 1.9997$

$x_3^{(6)} = 1.0019$

$x_1^{(7)} = 2.9989$

$x_2^{(7)} = 2.0016$

$x_3^{(7)} = 0.9993$

$x_1^{(8)} = 3.0011$

$x_2^{(8)} = 1.9991$

$x_3^{(8)} = 1.0002$

$x_1^{(9)} = 2.9994$

$x_2^{(9)} = 2.0003$

$x_3^{(9)} = 1$

$x_1^{(10)} = 3.0001$

$x_2^{(10)} = 1.9999$

$x_3^{(10)} = 1$

$x_1^{(11)} = 2.9999$

$x_2^{(11)} = 2$

$x_3^{(11)} = 1$

$x_1^{(12)} = 3$

$x_2^{(12)} = 2$

$x_3^{(12)} = 1$

$x_1^{(13)} = 3$

$x_2^{(13)} = 2$

$x_3^{(13)} = 1$

Hence the solution of the given system of linear equation is :

$x_1 = 3$, $x_2 = 2$, $x_3 = 1$

□ □ □

Chapter-13

Forward Difference

Q.1. Construct a forward difference table for the following given data.

x	**3.60**	**3.65**	**3.70**	**3.75**
y	**36.598**	**38.475**	**40.447**	**42.521**

Ans.:

x	**y**	**Δy**	**$\Delta^2 y$**	**$\Delta^3 y$**
3.60	36.598	1.877	0.095	0.007
3.65	38.475	1.972	0.102	
3.70	40.447	2.074		
3.75	42.521			

□ □ □

Chapter-14

Backward Difference

Q.1. Construct a backward difference table form the following data :

sin 30° = 0.5000, sin 35° = 0.5736, sin 40° = 0.6428, sin 45° = 0.7071

Assuming third difference to be constant find the value of sin 25°.

Ans.:

x	**y**	**∇y**	**∇^2y**	**∇^3y**
25	?			
		∇y_{30} =?		
30	0.5000		$\nabla^2 y_{35}$ = ?	
		0.0736		$\nabla^3 y_{40}$ = ?
35	0.5736		-0.0044	
		0.0692		-0.0005
40	0.6428		-0.0049	
		0.0643		
45	0.7071			

Since we know that ∇^3y should be constant so

$$\nabla^3 y_{40} = -0.0005$$

$$\Rightarrow \nabla^2 y_{40} - \nabla^2 y_{35} = -0.0005$$

$$\Rightarrow -0.0044 - \nabla^2 y_{35} = -0.0005$$

$\nabla^2 y_{35} = +0.0005 - 0.0044$

$= -0.0039$

Again $\nabla^2 y_{35} = -0.0039$

$\nabla y_{35} - \nabla y_{30} = -0.0039$

$\Rightarrow 0.0736 - \nabla y_{30} = -0.0039$

$\nabla y_{30} = 0.0039 + 0.0736$

$= 0.0775$

Again $\nabla y_{30} = 0.0775$

$y_{30} - y_{25} = 0.0775$

$\Rightarrow 0.5000 - y_{25} = 0.0775$

$y_{25} = 0.5000 - 0.0775$

$= 0.4225$

Hence $\sin 25° = 0.4225$

□ □ □

Chapter-15

Newton Gregory Formula for Forward Interpolation

Q.1. Use Newton formula for interpolation to find the net premium at the age 25 from the table given below :

Age	**20**	**24**	**28**	**32**
Annual net premium	**0.01427**	**0.01581**	**0.01772**	**0.01996**

Ans.:

Age (x)	**$f(x)$**	**$\Delta f(x)$**	**$\Delta^2 f(x)$**	**$\Delta^3 f(x)$**
20	0.01427			
		0.00154		
24	0.01581		0.00037	
		0.00191		-0.00004
28	0.01772		0.00033	
		0.00224		
32	0.01996			

Here $a = 20$, $h = 4$ and $x = a + hu$

$\Rightarrow x = a + hu$

$25 = 20 + 4 \times u$

$5 = 4u \Rightarrow u = 1.25$

Using following Newton's Gregory forward interpolation formula :

$$f(a + hu) = f(a) + u^{(1)}\,\Delta f(a) + \frac{u^{(2)}}{2!}\,\Delta^2 f(a) + \frac{u^{(3)}}{3!}\,\Delta^3 f(a) + \_\_\_\_\_$$

$$\Rightarrow f(25) = 0.01427 + 1.25\,(0.00154) + \frac{1.25\,(0.25)}{1 \times 2}\,(0.00037) + \frac{1.25\,(0.25)\,(-0.75)}{1 \times 2 \times 3}\,(-0.00004)$$

$$\Rightarrow f(25) = 0.01427 + 0.001925 + 0.0000578 + 0.0000015 = 0.0162543$$

Q.2. From the following table find the number of students who obtained less than 45 marks :

Marks	No. of Students
30 – 40	**31**
40 – 50	**42**
50 – 60	**51**
60 – 70	**35**
70 – 80	**31**

Ans.:

Marks (x)	No. of Students $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
Less than 40	31				
		42			
Less than 50	73		9		
		51		-25	
Less than 60	124		-16		37
		35		12	
Less than 70	159		-4		
		31			
Less than 80	190				

Here $a = 40$, $h = 10$ and $a + hu = 45$

$$\Rightarrow 40 + 10 \times u = 45$$

$10u = 5$

$u = \frac{1}{2}$

using following forward interpolation formula :

$$f(x) = f(a) + u^{(1)}\,\Delta f(a) + \frac{u^{(2)}}{2!}\,\Delta^2 f(a) + \frac{u^{(3)}}{3!}\,\Delta^3 f(a) + \_\_\_\_\_$$

$$\therefore\ f(45) = f(40) + \frac{1}{2}\,\Delta f(40) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\,\Delta^2 f(40) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\,\Delta^3 f(40)$$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}\,\Delta^4 f(40)$$

$$= 31 + \frac{1}{2}(42) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1 \times 2}(9) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \times 2 \times 3}(-25) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \times 2 \times 3 \times 4}(37)$$

$= 31 + 21 - 1.125 - 1.5625 - 1.4453$

$= 47.8672 = 48$ (approximately)

Hence the no. of students who obtained less than 45 marks are 48.

Q.3. Find the cubic polynomial which takes the following values

x	**0**	**1**	**2**	**3**
$f(x)$	**1**	**0**	**1**	**10**

Find $f(4)$

Ans.: Here we know that a = 0, h = 1 then form Newton's Gregory forward interpolation formula.

$$P_n(x) = f(0) + {}^{x}c_1\,\Delta f(0) + {}^{x}c_2\,\Delta^2 f(0) + \_\ \_\_\ {}^{x}c_n\,\Delta^n f(0) \qquad \_\_\_\_ (1)$$

(x)	f(x)	Δf(x)	$\Delta^2 f$(x)	$\Delta^3 f$(x)
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		$f(4) - 27 = 6$
3	10		$f(4) - 19$	(it should be constant)
		$f(4) - 10$		
4	$f(4)$			

Substituting the values in equation (1) from above table :

$$P_3(x) = 1 + x(-1) + \frac{x(x-1)}{1 \times 2}(2) + \frac{x(x-1)(x-2)}{1 \times 2 \times 3}(6)$$

$$P_3(x) = 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$= x^3 - 2x^2 + 1$$

Hence the required polynomial of degree three is

$$x^3 - 2x^2 + 1$$

Again $f(4) - 27 = 6$

$$\Rightarrow f(4) = 33$$

□ □ □

Chapter-16

Newton's Formula for Backward Interpolation

Q.1. **The population of a town in decennial census was as given below :**

Year	**1891**	**1901**	**1911**	**1921**	**1931**
Population (in thousands)	**46**	**66**	**81**	**93**	**101**

Estimate the population for the year 1925.

Ans.:

Year (x)	**Population (in thousand) $f(x)$**	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here x = 1925, h = 10 , a = 1891 and a + nh = 1931

$\therefore$ (a + hn) + uh = 1925

1931 + uh = 1925

$$uh = \frac{1925 - 1931}{10} = -0.6$$

Now using Newton's Backward interpolation formula :

$$f(a + nh + uh) = f(a + nh) + \frac{u}{1!}\nabla f(a + nh) + \_\_\_\_ + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a + nh)$$

$$f(1925) = 101 + (-0.6) \times 8 + \frac{(-0.6)(0.4)}{2!}(-4) + \frac{(-0.6)(0.4)(1.4)}{3!}(-1) + \frac{(-0.6)(0.4)(1.4)(2.4)}{4!} \times (-3)$$

$$= 101 - 4.8 + 0.48 + 0.056 - 0.1008$$

= 96.6352 thousand (approximately)

□ □ □

Chapter-17

Divided Difference Interpolation

Q1. Construct a divided difference table from the following data :

x	**1**	**2**	**4**	**7**	**12**
f(x)	**22**	**30**	**82**	**106**	**216**

Ans.:

x	**f(x)**	**$\Delta f(x)$**	**$\Delta^2 f(x)$**	**$\Delta^3 f(x)$**	**$\Delta^4 f(x)$**
1	22				
		$\frac{30-22}{2-1} = 8$			
2	30		$\frac{26-8}{4-1} = 6$		
		$\frac{82-30}{4-2} = 26$		$\frac{(-3.6-6)}{7-1}$ = -1.6	
4	82		$\frac{8-26}{7-2}$ = -3.6		$\frac{0.535-(-1.6)}{12-1}$ = 0.194
		$\frac{106-82}{7-4} = 8$		$\frac{1.75-(-3.6)}{12-2}$ = 0.535	

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
7	106		$\frac{22-8}{12-4} = 1.75$		
		$\frac{216-106}{12-7}$ $= 22$			
12	216				

Q.2. By means of Newton's divided difference formula find the value of $f(2)$, $f(8)$ and $f(15)$ from the following table :

x	**4**	**5**	**7**	**10**	**11**	**13**
$f(x)$	**48**	**100**	**294**	**900**	**1210**	**2028**

Ans.: Newton's divided difference formula for 4, 5, 7, 10, 11, 13 is :

$$f(x) = f(4) + (x-4)\underset{5}{\Delta} f(x) + (x-4)(x-5)\underset{5,7}{\Delta^2} f(4) + (x-4)(x-5)(x-7)\underset{5,7,10}{\Delta^3} f(4)$$

$$+ (x-4)(x-5)(x-7)(x-10)\underset{5,7,10,11}{\Delta^4} f(4) + \_\_\_\_\_\_\_ \qquad \_\_\_ (1)$$

So constructing the following divided difference table :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100		$\frac{97-52}{7-4} = 15$		
		$\frac{294-100}{7-4} = 97$		$\frac{21-15}{10-4} = 1$	

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
7	294		$\frac{202-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

Substituting the values from above table in equation (1)

$f(x) = 48 + 52 (x - 4) + 15 (x - 4) (x - 5) + (x - 4) (x - 5) (x - 7)$

$= x^2 (x - 1)$ _ _ _(2)

Now substituting x = 2, 8 and 15 in equation (2)

$f(2) = 4 (2 -1) = 4$

$f(8) = 64 (8 - 1) = 448$

$f(15) = 225 (15 - 1) = 3150$

Q.3. Find the polynomial of the lowest possible degree which assumes the values 3, 12, 15, -21 when x has values 3, 2, 1, -1 respectively.

Ans.: Constructing table according to given data

x	f**(x)**	Δf**(x)**	$\Delta^2 f$**(x)**	$\Delta^3 f$**(x)**
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

Substituting the values in Newton's divided difference formula :

$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1) \_\ \_\ \_ (x - x_{n-1}) + f(x_0, x_1, x_2 \,..x_n)$

$= -21 + \{x - (-1)\}\, 18 + \{ x - (-1)\}(x - 1)(-7) + \{x - (-1)\}(x - 1)(x - 2)(1)$

$= x^3 - 9x^2 + 17x + 6$

□ □ □

Chapter-18

Lagrange's Interpolation

Q.1. Given that

$f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(4) = 16$, $f(7) = 128$

Find the value of $f(5)$ with the help of Lagrange's interpolation formula.

Ans.: According to question

$x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 7$, and

$f(x_0) = 2$, $f(x_1) = 4$, $f(x_2) = 8$, $f(x_3) = 16$, and $f(x_4) = 128$,

Using Lagrange's formula for x = 5

$$f(5) = \frac{(5-2)(5-3)(5-4)(5-7)}{(1-2)(1-3)(1-4)(1-7)}\times 2 + \frac{(5-1)(5-3)(5-4)(5-7)}{(2-1)(2-3)(2-4)(2-7)}\times 4$$

$$+ \frac{(5-1)(5-2)(5-4)(5-7)}{(3-1)(3-2)(3-4)(3-7)}\times 8 + \frac{(5-1)(5-2)(5-3)(5-7)}{(4-1)(4-2)(4-3)(4-7)}\times 16$$

$$+ \frac{(5-1)(5-2)(5-3)(5-4)}{(7-1)(7-2)(7-3)(7-4)}\times 128$$

$$= \frac{-2}{3} + \frac{32}{5} - 24 + \frac{128}{3} + \frac{128}{15} = \frac{494}{15}$$

$$= 32.93333$$

Hence $f(5) = 32.9333$

Q.2. Find the form of function given by the following table :

x	**3**	**2**	**1**	**-1**
$f(x)$	**3**	**12**	**15**	**-21**

Ans.: According to question

$x_0 = 3$, $x_1 = 2$, $x_2 = 1$ and $x_3 = -1$

$f(x_0) = 2$, $f(x_1) = 12$, $f(x_2) = 15$ and $f(x_3) = -21$

Now substituting above values in Lagrange's formula :

$$f(x) = \frac{(x-2)(x-1)(x+1)}{(3-2)(3-1)(3+1)} \times 3 + \frac{(x-3)(x-1)(x+1)}{(2-3)(2-1)(2+1)} \times 12$$

$$+ \frac{(x-3)(x-2)(x+1)}{(1-3)(1-2)(1+1)} \times 15 + \frac{(x-3)(x-2)(x-1)}{(-1-3)(-1-2)(-1-1)} \times -21$$

$$= \frac{3}{8}(x^3 - 2x^2 - x + 2) - 4(x^3 - 3x^2 - x + 3) + \frac{15}{4}(x^3 - 4x^2 + x + 6)$$

$$+ \frac{7}{8}(x^3 - 6x^2 + 11x - 6)$$

$$f(x) = x^3 - 9x^2 + 17x + 6$$

Q.3. By means of Lagrange's formula prove that :

$$\mathbf{y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]}$$

Ans.: Here we are given y_{-3}, y_{-1} y_1 and y_3 and we have to evaluate y_0.

Using Lagrange's formula

$$y_0 = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)} y_{-1}$$

$$+ \frac{(0+3)(0+1)(0-3)}{(1+3)(1+1)(1-3)} y_1 + \frac{(0+3)(0+1)(0-3)}{(3+3)(3+1)(3-1)} y_3$$

$$= \frac{-1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}(y_3 - y_1) - (y_{-1} - y_{-3})$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

Hence proved.

□ □ □

Chapter-19

Spline Interpolation

Q.1. Given the set of data points (1, – 8), (2, – 1) and (3, 18) satisfying the function y = f(x). find the linear splines satisfying the given data. Determine the approximate values of y(2.5) and y^1(2.0).

Ans.: Let the given points be A (1, – 8), B (2, – 1) and C (3, 18) equation of AB is

s_1 (x) = -8 + (x – 1) 7 [$s_i(x) = y_{i-1} + m_i (x - x_{i-1})$]

= -8 + 7x – 7

= 7x -15

And equation of BC is

s_2 (x) = -1 + (x – 2) (19)

= -1 + 19x -38

= 19x – 39 _ _ _ (1)

Since x = 2.5 belongs to the interval [2, 3] we have

y (2.5) = s_2 (2.5) = -19(2.5) -39 = 8.5

And y^1(x) = +19 [from equation (1)]

Here we note that the splines s_i (x) are continuous in the interval [1, 3] but their slopes are discontinuous.

□ □ □

Chapter-20

Quadratic Splines

Q.1. Given the set of data points (1, – 8), (2, – 1) and (3, 18) satisfying the function y = *f*(x). find the quadratic splines satisfying the given data. Find also the approximate values of y(2.5) and y' (2.0).

Ans.: Since we know that

$$m_{i-1} + m_i = \frac{2}{h_i}(y_i - y_{i-1}) \qquad [i = 1, 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots n]$$

we have h = 1

taking i = 1

$$m_0 + m_1 = 14$$

taking i = 2

$$m_1 + m_2 = 38$$

Since $m_0 = m_1$ we obtain $m_0 = m_1 = 7$ and $m_2 = 31$ using following equation

$$s_i(x) = \frac{1}{h_i}\left[-\frac{(x_i - x)^2}{2}m_{i-1} + \frac{(x - x_{i-1})^2}{2}m_i\right] + y_{i-1} + \frac{h_i}{2}m_{i-1}$$

$$s_2(x) = -\frac{(x_2 - x)^2}{2}m_1 + \frac{(x - x_1)^2}{2}m_2 + y_1 + \frac{1}{2}m_1$$

$$= -\frac{(3 - x)^2}{2}\times(7) + \frac{(x - 2)^2}{2}(31) - 1 + \frac{7}{2}$$

$$= -\frac{(3-x)^2}{2}(7) + \frac{31}{2}(x-2)^2 + \frac{5}{2}$$

$= 12x^2 - 41x + 33$

Since 2.5 lies in the interval [2, 3]

Hence

$y(2.5) = s_2(2.5)$

$= 12(2.5)^2 - 41(2.5) + 33$

$= 12 \times 6.25 - 41 \times 2.5 + 33$

$= 5.5$

$y'(x) = 24x - 41$

$y'(2) = 24 \times 2 - 41$

$= 48 - 41$

$= 7.0$

□ □ □

Chapter-21

Cubic Splines

Q.1. Given the set of data points (1, – 8), (2, – 1) and (3, 18) satisfying the function y = f(x). find the cubic splines satisfying the given data. Determine the approximate values of y (2.5) and y' (2.0).

Ans.: We have n = 2 and $p_0 = p_2 = 0$ therefore from the following relation :

$$p_{i-1} + 4p_i + p_{i+1} = \frac{6}{h^2}(y_{i+1} - 2y_i + y_{i-2}) \qquad (i = 1, 2......n-1)$$

gives

$$p_1 = 18$$

If s_1 (x) and s_2 (x) are respectively, the cubic splines in the intervals $1 \le x \le 2$ and $2 \le x \le 3$, we obtain

$$s_1(x) = 3(x - 1)^3 - 8(2 - x) - 4(x - 1)$$

and $$s_2(x) = 3(3 - x)^3 + 22x - 48$$

We therefore have

$$y(2.5) = s_2(2.5) = \frac{3}{8} + 7 = 7.375$$

and $$y^1(2.0) = s_2(2.0) = 13.0$$

□ □ □

Chapter-22

Numerical Differentiation

Q.1. From the following table of values of x and y obtain dy/dx and d^2y/dx^2 at x = 1.1.

x	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**
y	**0.00**	**0.1280**	**0.5440**	**1.2960**	**2.4320**	**4.00**

Ans.: According to given question,

h = 0.2, a = 1 and x = 1.1

Here 1.1 is close to the initial value so using Newton-Gregory forward difference formula.

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0.00				
		0.1280			
1.2	0.1280		0.2880		
		0.4160		0.0480	
1.4	0.5440		0.3360		0
		0.7520		0.0480	
1.6	1.2960		0.3840		0
		1.1360		0.0480	
1.8	2.4320		0.4320		
		1.5680			
2.0	4.0000				

Newton's Gregory forward formula is :

$$f(a + xh) = f(a) + {}^{x}c_1\, \Delta f(a) + {}^{x}c_2\, \Delta^2 f(a) + {}^{x}c_3\, \Delta^3 f(a) + \ldots\ldots..$$

or $$f(a + xh) = f(a) + x\,\Delta f(a) + \frac{x^2 - x}{2}\,\Delta^2 f(a) + \frac{x^3 - 3x^2 + 2x}{6}\,\Delta^3 f(a) + \ldots \quad \text{---} \ (1)$$

Differentiating both sides of the equation (1) w. r. t. x

$$hf'(a + xh) = \Delta f(a) + \frac{(2x-1)}{2}\,\Delta^2 f(a) + \frac{3x^2 - 6x + 2}{6}\,\Delta^3 f(a) + \ldots\ldots \quad \text{---} \ (2)$$

Again differentiating equation (2) w. r.t. x

$$h^2 f'(a + xh) = \Delta^2 f(a) + (x - 1)\,\Delta^3 f(a) + \ldots\ldots. \quad \text{---} \ (3)$$

Here we have to find f' (1.1) and f'' (1.1)

Substituting a = 1, h = 0.2 and x = $\frac{1}{2}$ in equation (2) and (3)

$$0.2f'(1.1) = 0.1280 + 0 + \frac{1}{6}\left(3 \times \frac{1}{4} - 6 \times \frac{1}{2} + 2\right)(0.0480) + 0$$

Hence f' (1.1) = 0.630 --- (4)

And $$(0.2)^2 f''(1.1) = 0.2880 + \left(\frac{1}{2} - 1\right)(0.0480) + 0 = 0.264$$

Hence f'' (1.1) = 6.60 --- (5)

Q.2. Using divided difference find the value of f' (8) given that :

x	**6**	**7**	**9**	**12**
***f*(x)**	**1.556**	**1.690**	**1.908**	**2.158**

Ans.:

x	**y = *f*(x)**	**Δy**	**Δ²y**	**Δ³y**
$x_0 = 6$	1.556			
		0.134		

x	y = *f*(x)	Δy	Δ²y	Δ³y
$x_1 = 7$	1.690		- 0.0083	
		0.109		0.00051
$x_2 = 9$	1.908		-0.0052	
		0.083		
$x_3 = 12$	2.158			

Newton's divided difference formula is

$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)$
$\Delta^3 f(x_0) + (x + x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0) + \ldots\ldots$ ___ (1)

Differentiating both sides of equation (1) w. r.t. x

$f'(x) = \Delta f(x_0) + (2x - x_0 + x_1)\,\Delta^2 f(x_0) +$
$[3x^2 - 2x(x_0 + x_1 + x_2) + x_0x_1 + x_1x_2 + x_0x_2]\,\Delta^3 f(x_0)$ ___ (2)

Now substituting $x = 8$, $x_0 = 6$, $x_1 = 7$, $x_2 = 9$, $x_3 = 12$ in equation (2)

$f^1(8) = 0.134 + [2 \times 8 - 6 - 7]\,(-0.0083) +$
$[3 \times 64 - 2 \times 8\,(6 + 7 + 9) + 6 \times 7 + 7 \times 9 + 6 \times 9](0.00051)$
$= 0.134 - 0.0249 + (192 - 352 + 159)\,(0.00051)$
$= 0.10859$

□ □ □

Chapter-23

Numerical Integration

Q.1. Compute the value of following integral by Trapezoidal rule.

$$\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx$$

Ans.: Dividing the range of integration in equal intervals in the interval [0.2, 1.4]

$$\frac{1.4-0.2}{6} = \frac{1.2}{6} = 0.2 = h$$

x	sin x	$\log_e x$	e^x	$y = \sin x - \log + e^x$
0.2	0.19867	-1.6095	1.2214	y_0 = 3.0296
0.4	0.3894	-0.9163	1.4918	y_1 = 2.7975
0.6	0.5646	-0.5108	1.8221	y_2 = 2.8975
0.8	0.7174	-0.2232	2.2255	y_3 = 3.1661
1.0	0.8415	0.0000	2.7183	y_4 = 3.5598
1.2	0.9320	0.1823	3.3201	y_5 = 4.0698
1.4	0.9855	0.3365	4.0552	y_6 = 4.7042

Using following trapezoidal rule

$$I = \int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx$$

$= \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$

$= \frac{0.2}{2}[7.7338 + 2(16.4907)]$

$= 4.07152$

Q.2. Calculate the value of the integral $\int_4^{5.2} \log_e x \, dx$ by Simpson's '$\frac{1}{3}$' rule.

Ans.: First of all dividing the interval [4 5.2] in equal parts.

$\frac{5.2-4}{6} = \frac{1.2}{6} = 0.2 = h$

x_i	$y_i = \log_e x = \log_{10} x \times 2.30258$
4.0	$y_0 = 1.3862944$
4.2	$y_1 = 1.4350845$
4.4	$y_2 = 1.4816045$
4.6	$y_3 = 1.5260563$
4.8	$y_4 = 1.5686159$
5.0	$y_5 = 1.6049379$
5.2	$y_6 = 1.6486586$

Using following Simpson's '$\frac{1}{3}$' rule :

$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$

$= \frac{0.2}{3}[3.034953 + 18.232315 + 6.1004408]$

$= \frac{0.2}{3}[27.417709] = 1.8278472$

Q.3. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's '$\frac{3}{8}$' rule :

Ans.: Dividing the interval [0, 1] into six equal intervals.

$$h = \frac{1-0}{6} = \frac{1}{6}$$

x	**y =** $h = \frac{1}{(1+\mathbf{x}^2)}$
$x_0 = 0$	$y_0 = 1.000$
$x_0 + h = 1/6$	$y_1 = (36/37) = 0.97297$
$x_0 + 2h = 2/6$	$y_2 = (36/40) = 0.90000$
$x_0 + 3h = 3/6$	$y_3 = (36/45) = 0.80000$
$x_0 + 4h = 4/6$	$y_4 = (36/52) = 0.69231$
$x_0 + 5h = 5/6$	$y_1 = (36/61) = 0.59016$
$x_0 + 6h = 1$	$y_6 = (1/2) = 0.50000$

Using following Simpson's '3/8' rule.

$$\int_{x_0}^{x_0+nh} ydx = \frac{3h}{8} \ (y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ...) + 2(y_3 + y_6 + ...)$$

$$\int_0^1 ydx = \frac{1}{16} \ (1+ 0.5) + 3\ (0.97297 + 0.9 + 0.69231 + 0.59016) + 2\ (0.8)$$

$$= \frac{1}{16} \ [1.5 + 9.46632 + 1.6] \quad = 0.785395$$

□ □ □

Chapter-24

Numerical Solution for Differential Equations [Euler's Method]

Q.1. Use Euler's Method to determine an approximate value of y at x = 0.2 from initial value problem $\frac{dx}{dy} = 1 - x + 4y$ y(0) = 1 taking the step size h = 0.1.

Ans.: Here $h = 0.1$, $n = 2$, $x_0 = 0$, $y_0 = 1$

Given $\frac{dx}{dy} = 1 - x + 4y$

Hence $y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.1\,[1 - x_0 + 4y_0]$

$= 1 + 0.1\,[1 - 0 + 4 \times 1]$

$= 1 + 0.1\,[1 + 4]$

$= 1 + 0.5 \times 5$

$= 1.5$

Similarly $y_2 = y_1 + hf(x_0 + h, y_1)$

$= 1.5 + 0.1[1 - 0.1 + 4 \times 1.5]$

$= 2.19$

Q.2. Using Euler's Method with step-size 0.1 find the value of y(0.5) from the following differential equation $\frac{dx}{dy} = x^2 + y^2$, y(0) = 0

Ans.: Here h = 0.1, n = 5, $x_0 = 0$, $y_0 = 0$ and $f(x, y) = x^2 + y^2$

Hence $y_1 = y_0 + hf(x_0, y_0)$

$= 0 + (0.1)\ [0^2 + 0^2]$

$= 0$

Similarly $y_2 = y_1 + hf(x_0 + h, y_1)$

$= 0 + (0.1)\ [(0.1)^2 + 0^2]$

$= (0.1)^3$

$= 0.001$

$y_3 = y_2 + hf[x_0 + 2h, y_2]$

$= 0.001 + (0.1)\ [(0.2)^2 + (0.001)^2]$

$= 0.001 + 0.1\ [0.04 + 0.00001]$

$= 0.001 + 0.1\ [0.040001]$

$= 0.005$

$y_4 = y_3 + hf[x_0 + 3h, y_3]$

$= 0.005 + (0.1)\ [(0.3)^2 + (0.005)^2]$

$= 0.005 + (0.1)\ [0.09 + 0.000025]$

$= 0.014$

$y_5 = y_4 + hf[x_0 + 4h, y_4]$

$= 0.014 + (0.1)\ [(0.4)^2 + (0.014)^2]$

$= 0.014 + (0.1)\ [0.16 + 0.00196]$

$= 0.031$

Hence the required solution is 0.031

□ □ □

Chapter-25

Numerical Solution for Differential Equations [Euler's Modified Method]

Q.1. Using Euler's modified method, obtain a solution of the equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with initial conditions y = 1 at x = 0 for the range $0 \le x \le 0.6$ in the step of 0.2. Correct upto four place of decimals.

Ans.: Here $f(x, y) = x + |\sqrt{y}|$

$x_0 = 0$, $y_0 = 1$, $h = 0.2$ and $x_n = x_0 + nh$

(i) At $x = 0.2$

First approximate value of y_1

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.2)[0 + 1]$$

$$= 1.2$$

Second approximate value of y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}[(0 + 1) + \{0.2 + \sqrt{1.2}\}]$$

$$= 1.2295$$

Third approximate value of y_1

$$y_1^{(3)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(2)})\}$$

$$= 1 + \frac{0.2}{2}\,[(0 + 1) + \{0.2 + \sqrt{1.2295}\,\}]$$

$$= 1 + 0.1\,[1 + 1.30882821]$$

$$= 1.2309$$

Fourth approximate value of y_1

$$y_1^{(4)} = y_0 + \frac{h}{2}\,\{f(x_0, y_0) + f(x_1, y_1^{(3)})\}$$

$$= 1 + \frac{0.2}{2}\,[(0 + 1) + (0.2 + \sqrt{1.2309})]$$

$$= 1 + 0.1\,[1 + 1.30945]$$

$$= 1.2309$$

Since the value of $y_1^{(3)}$ and $y_1^{(4)}$ is same

Hence at $x_1 = 0.2$, $y_1 = 1.2309$

(ii) At $x = 0.4$

First approximate value of y_2

$$y_2^{(1)} = y_1 + hf(x_1, y_1)$$

$$= 1.2309 + (0.2)\,\{0.2 + \sqrt{1.2309}\}$$

$$= 1.4927$$

Second approximate value of y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.2309 + \frac{0.2}{2}\,[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.4927}\,)]$$

$$= 1.2309 + 0.1\,[1.309459328 + (1.621761024]$$

$$= 1.5240$$

Third approximate value of y_2

$$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 1.2309 + \frac{0.2}{2} [(1.309459328 + (0.4 + \sqrt{1.5240}))]$$

$$= 1.2309 + 0.1 [1.309459328 + 1.634503949]$$

$$= 1.5253$$

Fourth approximate value of y_2

$$y_2^{(4)} = y_1 + \frac{h}{2} \{f(x_1, y_1) + f(x_2, y_2^{(3)})\}$$

$$= 1.2309 + \frac{0.2}{2} [(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.5253})]$$

$$= 1.2309 + 0.1 \{1.309459328 + 1.635030364]$$

$$= 1.5253$$

Hence at $x = 0.4$, $y_2 = 1.5253$

(ii) At $x = 0.6$

First approximate value of y_3

$$y_3^{(1)} = y_2 + hf(x_2, y_2)$$

$$= 1.5253 + 0.2 [0.4 + \sqrt{1.5253}]$$

$$= 1.8523$$

Second approximate value of y_3

$$y_3^{(2)} = y_2 + \frac{h}{2} \{f(x_2, y_2) + f(x_3, y_3^{(1)})\}$$

$$= 1.5253 + \frac{0.2}{2} [(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8523})]$$

$$= 1.8849$$

Third approximate value of y_3

$$y_3^{(3)} = y_2 + \frac{h}{2}\{f(x_2, y_2) + f(x_3, y_3^{(2)})\}$$

$$= 1.5253 + \frac{0.2}{2}[(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8849})]$$

$$= 1.8851$$

Fourth approximate value of y_3

$$y_3^{(4)} = y_2 + \frac{h}{2}\{f(x_2, y_2) + f(x_3, y_3^{(3)})\}$$

$$= 1.5253 + \frac{0.2}{2}[(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8851})]$$

$$= 1.8851$$

Hence at $x = 0.6$, $y_3 = 1.8851$

□ □ □

Chapter-26

Numerical Solution for Differential Equations [Runge – Kutta Method]

Q.1. Using Runge – Kutta method find an approximate value of y for x = 0.2 in step of 0.1 if $\frac{dy}{dx} = x + y^2$ given y = 1 when x = 0

Ans.: Here $f(x, y) = x + y^2$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

$K_1 = hf(x_0, y_0) = 0.1[0 + 1]$

$= 0.1$ ………. ___ (1)

$$K_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}K_1\right)$$

$$= 0.1\left[\left(0 + \frac{1}{2}(0.1)\right) + \left(1 + \frac{1}{2} \times 0.1152\right)^2\right]$$

$= 0.1152$ ___ (2)

$$K_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}K_2\right)$$

$$= 0.1\left[\left(0 + \frac{1}{2}(0.1)\right) + \left\{1 + \left(\frac{1}{2} \times 0.1152\right)\right\}^2\right]$$

$= 0.1168$ ___ (3)

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1\left[\; 0+0.1 \;+\; 1+0.1168 \;^{2}\right]$

$= 0.1347$ — — — (4)

and

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}\;\; 0.1+2(0.1152)+2(0.1168)+0.1347$ {using equation (1), (2), (3) and (4)}

$= 0.1165$

Hence $y_1 = y_0 + K = 1 + 0.1165$

$= 1.1165$ — — — (5)

Again $x_1 = x_0 + h = 0.1$, $y_1 = 1.1165$, $h = 0.1$

Now

$K_1 = hf(x_1, y_1)$

$= 0.1\left[0.1+(1.1165)^2\right]$

$= 0.1347$ — — — (6)

$K_2 = hf\left[x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_1\right]$

$= 0.1\left[\left\{0.1+\frac{1}{2}(0.1)\right\}+\left\{1.1165+\frac{1}{2}(0.1347)\right\}^2\right]$

$= 0.1551$ — — — (7)

$K_3 = hf\left[x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}K_2\right]$

$= 0.1\left[\left\{0.1+\frac{1}{2}(0.1)\right\}+\left\{1.1165+\frac{1}{2}(0.1551)\right\}^2\right]$

$= 0.1576$ — — — (8)

$K_4 = hf\ (x_1 + h, y_1 + K_3)$

$= (0.1)\left[(0.1 + 0.1) + (1.1165 + 0.1576)^2\right]$

$= 0.1823$ ––– (9)

and

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}\left[0.1347 + 2(0.1551) + 2(0.1576) + 0.1823\right]$ {using equation (6), (7), (8) and (9)}

$= 0.1570$

Hence

$y(0.2) = y_2 = y_1 + K$

$= 1.1165 + 0.1570$

$= 1.2735$

which is required solution.

Q.2. Use Runge-Kutta method to solve y' = x y for x = 1.4. Initially x = 1, y = 2 (tale h = 0.2).

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Ans.: (i) Here $f(x, y) = xy$, $x_0 = 1$, $y_0 = 2$, $h = 0.2$

$K_1 = hf(x_0, y_0)$

$= 0.2[1 \times 2]$

$= 0.4$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$= 0.2\left[\left(1 + \frac{0.2}{2}\right) \times \left(2 + \frac{0.4}{2}\right)\right]$

$= 0.2\left[\ 1+0.1\ \times\ 2+0.2\ \right]$

$= 0.2\left[\ 1.1\quad 2.2\ \right]$

$= 0.484$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2})$

$= 0.2\left[\left(1+\frac{0.2}{2}\right)x\left(2+\frac{0.484}{2}\right)\right]$

$= 0.49324$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.2\left[\ 1+0.2\ \times\ 2+0.49324\ \right]$

$= 0.5983776$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}\ 0.4+2(0.484)+2(0.49324)+0.5983776$

$= 0.4921429$

$y_1 = y_0 + K$

$= 2 + 0.4921429$

$= 2.4921429$

(ii) $x_1 = x_0 + h = 1 + 0.2 = 1.2$, $y_1 = 2.4921429$ and $h = 0.2$

$K_1 = hf(x_1, y_1)$

$= 0.2[(1.2)\ (2.4921429)]$

$= 0.5981143$

$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$

$$= 0.2\left[\left(1.2+\frac{0.2}{2}\right)\times\left(2.4921+\frac{0.5981143}{2}\right)\right]$$

$= 0.81824$

$$K_3 = hf(x_1+\frac{h}{2}, y_1+\frac{K_2}{2})$$

$$= 0.2\left[\left(1.2+\frac{0.2}{2}\right)\times\left(2.4921+\frac{0.81824}{2}\right)\right]$$

$= 0.7543283$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$$= 0.2\left[\ 1.2+0.2\ \times\ 2.4921+0.7543\ \right]$$

$= 0.9090119$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$= 0.7753$

$y_2 = y_1 + K$

$= 2.4921 + 0.7753$

$= 3.26752$

$\therefore\ y(1.4) = 3.26752$

□ □ □

Chapter-27

Boundary Valve Problem - Shooting Method

Q.1 Solve the Boundary Value Problem y'' (x) = y (x) ; y(0) = 0 ; y(1) = 1.1752 by the shooting method taking m_0 = 0.7 and m_1 = 0.8

Ans.: By Taylor's Series

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^{IV}(0) + \frac{x^5}{120}y^{V}(0) + \frac{x^6}{720}y^{VI}(0) + \ldots \quad \text{---} (1)$$

Since $y''(x) = y(x)$ we have

$y''(x) = y'(x)$

and $y^{IV}(x) = y''(x) = y(x)$

$y^{V}(x) = y'(x)$

$y^{VI}(x) = y''(x) = y(x)$ ………….

Putting x = 0 in above we get

$y''(0) = y(0) = 0$, $y'''(0) = y'(0)$

$y^{IV}(0) = 0$, $y''(0) = y'(0)$ …………

Substituting these values in equation (1)

$$y(x) = y^{I}(0)\left[x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^9}{362800} + \ldots\ldots\right]$$

Since $y(0) = 0$

Hence $y(1) = y'(0)\left[x + \frac{1}{6} + \frac{1}{120} + \frac{1}{5040} + \ldots\ldots\ldots\ldots\right]$

$= y'(0)(1.1752)$

_ _ _ (2)

With $y'(0) = m_0 = 0.7$

So equation (2) gives

$y(1) \approx 0.8226$

Similarly $y'(0) = m_0 = 0.8$ gives

$y(1) \approx 0.9402$

Using linear interpolation, we obtain

$$m_2 = 0.7 + (0.1)\ \frac{1.1752 - 0.8226}{0.9402 - 0.8226}$$

$= 0.9998$

Which is closer to exact value of $y'(0) = 1$ with this value of m_2, we solve the initial value problem $y''(x) = y(x)$, $y(0) = 0$, $y'(0) = m_2$

_ _ _ (3)

and continue the process as above until the value of y(1) is obtained to the desired accuracy.

□ □ □